Chapter Outline

12.1 Solving Systems of Linear Equations by Using Augmented Matrices
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Train Wheel Design

The axle of a train is solid so that the wheels must turn together. On the arc of a circular curve in the tracks, the wheels must travel different distances while turning the same number of times. This is accomplished by using a slightly slanted wheel. This allows the wheels to slide on the rails so that the point of contact with the rail varies the turning radius of each wheel. In the figure, which wheel is the inside wheel on a curve and which is the outside wheel? This question is examined in Exercise 61 of Exercises 12.1.
Section 12.1  Solving Systems of Linear Equations by Using Augmented Matrices

The laws of mathematics can be arrived at by the principle of looking for the simplest concepts and the link between them. —Réne Descartes (French mathematician and philosopher, 1596–1650)

Objective:
1. Use augmented matrices to solve systems of two linear equations with two variables.

In Chapter 3, we covered two algebraic methods for solving systems of linear equations: the substitution method and the addition method.

Why Do We Need Another Algebraic Method for Solving Systems of Linear Equations?
The answer to this question is that the substitution method and the addition method work fine for relatively simple systems, but these methods become increasingly complicated to execute for messy coefficients or for larger systems of equations. Graphs and tables are also limited to linear systems with only two variables.

Thus to handle larger systems of linear equations, we will develop over the next two sections the augmented matrix method. This method is very systematic and simplifies the solution of larger systems of linear equations. This method is also well suited to implementation by computer or calculator. After we have developed the augmented matrix method in this section, we illustrate in Section 12.2 how to use the TI-84 Plus calculator to implement this method.

1. Use Augmented Matrices to Solve Systems of Two Linear Equations with Two Variables

We now develop the augmented matrix method, a method that has many similarities to the addition method. A matrix is a rectangular array of numbers arranged into rows and columns. Each row consists of entries arranged horizontally. Each column consists of entries arranged vertically. The dimension of a matrix is given by stating first the number of rows and then the number of columns. The dimension of the following matrix is $2 \times 3$.

$$\begin{bmatrix} 5 & 6 & -1 \\ 4 & 3 & 1 \end{bmatrix}$$

The entries in an augmented matrix for a system of linear equations consist of the coefficients and constants in the equations. To form the augmented matrix for a system, first we align the similar variables on the left side of each equation and the constants on the right side. Then we form each row in the matrix from the coefficients and the constant of the corresponding equation. A 0 should be written in any position that corresponds to a missing variable in an equation.

System of Linear Equations
$$\begin{cases} 4x + y = 3 \\ 3x - 2y = 16 \end{cases}$$

Augmented Matrix
$$\begin{bmatrix} 4 & 1 & 3 \\ 3 & -2 & 16 \end{bmatrix}$$

A Mathematical Note
James Joseph Sylvester (1814–1897) wrote a paper formulating many of the properties of matrices in 1850 and introduced the term matrix.
To solve systems of linear equations by using augmented matrices, we need to be able to represent these systems by using matrices and also be able to write the system of linear equations represented by a given augmented matrix.

**Example 1**  Writing Augmented Matrices for Systems of Linear Equations

Write an augmented matrix for each system of linear equations.

**Solution**

(a) \[ \begin{align*} 2x - 5y &= 11 \\ 3x + 4y &= 5 \end{align*} \]

The \(x\)-coefficients 2 and 3 are in the first column. The \(y\)-coefficients \(-5\) and \(4\) are in the second column. The constants \(11\) and \(5\) are in the third column.

(b) \[ \begin{align*} 2x + 3y &= 29 \\ 5x - 4y + 8 &= 0 \end{align*} \]

First write each equation with the constant on the right side of the equation. Then form the augmented matrix.

(c) \[ \begin{align*} 2x - 4 &= 0 \\ 6x + y &= 4 \end{align*} \]

Write each equation in the form \(Ax + By = C\), using zero coefficients as needed. Then form the augmented matrix.

**Self-Check 1**

Write the augmented matrix for the system \[ \begin{align*} 5x + y &= -3 \\ 3x + 2y &= 8 \end{align*} \].

To solve systems of linear equations by using augmented matrices, we need to be able to represent these systems by using matrices and also be able to write the system of linear equations represented by a given augmented matrix.

**Example 2**  Writing Systems of Linear Equations Represented by Augmented Matrices

Using the variables \(x\) and \(y\), write a system of linear equations that is represented by each augmented matrix.

**Solution**

(a) \[ \begin{bmatrix} 4 & 5 & 17 \\ 5 & -3 & 12 \end{bmatrix} \]

Each row of the matrix represents an equation with an \(x\)-coefficient, a \(y\)-coefficient, and a constant on the right side of the equation. When the coefficient of a variable is 0, you do not have to write this variable in the equation. For example, \(0x + 2y = 10\) can be written as \(2y = 10\).

(b) \[ \begin{bmatrix} 8 & 3 & -1 \\ 0 & 2 & 10 \end{bmatrix} \]

The last pair of equations can be written as \(x + 0y = 7\) and \(0x + y = -8\). Thus the solution of this system is the ordered pair \((7, -8)\).

(c) \[ \begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -8 \end{bmatrix} \]

**Self-Check 2**

Using the variables \(x\) and \(y\), write the system of linear equations represented by the augmented matrix \[ \begin{bmatrix} 4 & -3 & 1 \\ 2 & 9 & 4 \end{bmatrix} \].
In order to simplify or use matrices with a calculator, we must first learn how to enter a matrix into a calculator. Technology Perspective 12.1.1 illustrates how to do this with a TI-84 Plus calculator. Note that the MATRIX feature is the secondary function of the $\mathbf{\text{2nd}}$ key.

**Technology Perspective 12.1.1**

Entering a Matrix into a Calculator

Enter the matrix from Example 2(a) into a calculator.

**TI-84 Plus Keystrokes**

1. Access the matrix EDIT menu by pressing $\mathbf{\text{2nd}}$ and then $\mathbf{\text{EDIT}}$.
2. Enter the dimensions of this $2 \times 3$ matrix by pressing $\mathbf{2}$ and then $\mathbf{3}$.
3. Enter each element of the matrix. Press $\mathbf{\text{ENTER}}$ after each value. Then press $\mathbf{\text{2nd}}$ and then $\mathbf{\text{QUIT}}$.
4. To display a matrix that has been entered as Matrix 1, press $\mathbf{\text{MATRX}}$, then $\mathbf{1}$.

**TI-84 Plus Screens**

The augmented matrix \[
\begin{bmatrix}
1 & 0 & 7 \\
0 & 1 & -8
\end{bmatrix}
\] in Example 2(c) is an excellent example of the type of matrices that we want to produce. It follows immediately from this matrix that the solution of the corresponding system of linear equations has $x = 7$ and $y = -8$. In ordered-pair notation, the answer is $(7, -8)$.

**What Matrix Represents the Solution of a System of Linear Equations?**

A matrix of the form \[
\begin{bmatrix}
1 & 0 & k_1 \\
0 & 1 & k_2
\end{bmatrix}
\] represents a system of linear equations whose solution is the ordered pair $(k_1, k_2)$. The material that follows shows how to produce this form.

**Equivalent systems** of equations have the same solution set. The strategy for solving a system of linear equations is to transform the system into an equivalent system composed of simpler equations. In Chapter 3, we used the properties of equality to justify the transformations we used to solve systems of linear equations. We review these transformations and give the corresponding elementary row operations on a matrix.
Transformations Resulting in Equivalent Systems

1. Any two equations in a system may be interchanged.
2. Both sides of any equation in a system may be multiplied by a nonzero constant.
3. Any equation in a system may be replaced by the sum of itself and a constant multiple of another equation in the system.

Elementary Row Operations on Augmented Matrices

1. Any two rows in the matrix may be interchanged.
2. Any row in the matrix may be multiplied by a nonzero constant.
3. Any row in the matrix may be replaced by the sum of itself and a constant multiple of another row.

We use the elementary row operations on an augmented matrix just as if the rows were the equations they represent. This is illustrated by the parallel development in Example 3.

Example 3  Solving a System by Using the Addition Method and the Augmented Matrix Method

Solve the system \[ \begin{cases} 3x + 2y = -2 \\ x - 3y = 14 \end{cases} \].

Solution

<table>
<thead>
<tr>
<th>Equation</th>
<th>Addition Method</th>
<th>Augmented Matrix Method</th>
</tr>
</thead>
</table>
| \[ \begin{cases} 3x + 2y = -2 \\ x - 3y = 14 \end{cases} \] | \[ \begin{cases} 3 & 2 & |-2 \\ 1 & -3 & |14 \end{cases} \] | \[ \begin{cases} 3 & 2 & |-2 \\ 1 & -3 & |14 \end{cases} \] Replace the first row with itself plus 3 times the second row. \[ r_1 \leftrightarrow r_2 \] This notation means that the first row is interchanged with the second row.
| \[ \begin{cases} x - 3y = 14 \\ 3x + 2y = -2 \end{cases} \] | \[ \begin{cases} 1 & -3 & |14 \\ 3 & 2 & |-2 \end{cases} \] Replace the second row with itself minus 3 times the first row. \[ r_2 = r_2 - 3r_1 \] \[ \begin{cases} x - 3y = 14 \\ 3x + 2y = -2 \end{cases} \] | \[ \begin{cases} 1 & -3 & |14 \\ 3 & 2 & |-2 \end{cases} \] Multiply the second row by \( \frac{1}{11} \). \[ r_2 = \frac{1}{11}r_2 \] \[ \begin{cases} x - 3y = 14 \\ y = -4 \end{cases} \] | \[ \begin{cases} 1 & -3 & |14 \\ 0 & 1 & |-4 \end{cases} \] Replace the first row with itself plus 3 times the second row. \[ r_1 = r_1 + 3r_2 \] \[ \begin{cases} x = 2 \\ y = -4 \end{cases} \] | \[ \begin{cases} 1 & 0 & |2 \\ 0 & 1 & |-4 \end{cases} \] This form gives the answer.

\[ x + 0y = \frac{2}{2} \]

Answer: \( (2, -4) \)

Self-Check 3

What is the solution of the system of linear equations represented by \[ \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 6 \end{bmatrix} \]?
Why Do We Use Arrows, Not Equal Symbols, to Denote the Flow from One Matrix to the Next?

The matrices are not equal because entries in them have been changed. However, if we use the elementary row operations, they do represent equivalent systems of equations and will help us to determine the solution of the system of linear equations.

The recommended first step in transforming an augmented matrix to the reduced form is to get a 1 to occur in the row 1, column 1 position.

Example 4 illustrates three ways to accomplish this. We suggest that you master the notation that is used to describe each step. This will help you better understand this topic and also prepare you to use calculator and computer commands to perform these elementary row operations.

Example 4  Using the Notation for Elementary Row Operations

Use the elementary row operations to transform \[
\begin{bmatrix}
3 & 1 & 2 \\
1 & 2 & 9
\end{bmatrix}
\]
into the form \[
\begin{bmatrix}
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\end{bmatrix}
\].

**Solution**

(a) By interchanging rows 1 and 2

\[
\begin{bmatrix}
3 & 1 & 2 \\
1 & 2 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 9 \\
3 & 1 & 2
\end{bmatrix}
\]

The notation \(r_1 \leftrightarrow r_2\) denotes that rows 1 and 2 have been interchanged.

(b) By multiplying row 1 by \(\frac{1}{3}\)

\[
\begin{bmatrix}
3 & 1 & 2 \\
1 & 2 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 2 \\
3 & 1 & 2
\end{bmatrix}
\]

\(r'_1\) denotes the new row 1 obtained by multiplying \(r_1\) by \(\frac{1}{3}\).

(c) By replacing row 1 with the sum of row 1 and \(-2\) times row 2.

\[
\begin{bmatrix}
3 & 1 & 2 \\
1 & 2 & 9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -3 & -16 \\
1 & 2 & 9
\end{bmatrix}
\]

\(r'_1\) denotes the new row 1 obtained by adding row 1 and \(-2\) times row 2.

Self-Check 4

Perform the indicated elementary row operations to produce new matrices.

a. \[
\begin{bmatrix}
4 & 1 & 11 \\
3 & 2 & 12
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\_ & \_ & \_ \\
3 & 2 & 12
\end{bmatrix}
\]

b. \[
\begin{bmatrix}
1 & 1 & 9 \\
2 & 3 & 23
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 9 \\
\_ & \_ & \_ \\
\_ & \_ & \_ \\
\end{bmatrix}
\]

c. \[
\begin{bmatrix}
1 & 3 & 7 \\
0 & 2 & 6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\_ & \_ & \_ \\
1 & 3 & 7
\end{bmatrix}
\]

The method used in Example 4(a) may be the easiest to apply, but it works only when there is a coefficient of 1 in another row to shift to this first row. The method used in part (c) is often used to avoid the fractions that can result from the method in part (b).

Example 5 uses the elementary row operations to solve a system of linear equations.
Example 5  Solving a System of Linear Equations by Using an Augmented Matrix

Use an augmented matrix and elementary row operations to solve \[
\begin{align*}
2x + 3y &= 4 \\
x + 4y &= -3
\end{align*}
\].

Solution

First form the augmented matrix.

\[
\begin{bmatrix}
2 & 3 & 4 \\
1 & 4 & -3
\end{bmatrix}
\]

To work toward the reduced form \[
\begin{bmatrix}
1 & 0 & k_1 \\
0 & 1 & k_2
\end{bmatrix}
\], interchange rows 1 and 2 to place a 1 in row 1, column 1.

Next add \(-2r_1\) to \(r_2\) to produce a 0 in row 2, column 1.

To complete the transformation to the form \[
\begin{bmatrix}
1 & 0 & k_1 \\
0 & 1 & k_2
\end{bmatrix}
\], produce a 0 in row 1, column 2 by adding \(-4r_2\) to \(r_1\).

Write the equivalent system of equations for the reduced form.

\[
\begin{align*}
x + 0y &= 5 \\
0x + y &= -2
\end{align*}
\]

Answer: \((5, -2)\)

Does this answer check?

Self-Check 5

Use an augmented matrix and elementary row operations to solve \[
\begin{align*}
2x + 9y &= -7 \\
x + 5y &= -3
\end{align*}
\].

Note that in Example 5, we first worked on column 1 and then on column 2. This column-by-column strategy is recommended for transforming all augmented matrices to reduced form. This strategy also is used in Example 6.

Example 6  Solving a System of Linear Equations by Using an Augmented Matrix

Use an augmented matrix and elementary row operations to solve \[
\begin{align*}
4x + 3y &= 3 \\
2x + 9y &= 4
\end{align*}
\].

Solution

First form the augmented matrix.

\[
\begin{bmatrix}
4 & 3 \\
2 & 9
\end{bmatrix}
\]

Then work on putting column 1 in the form \[
\begin{bmatrix}
1 & -k_1 \\
0 & -k_2
\end{bmatrix}
\].

\[
r_1 = r_1 \quad \text{means to multiply row 1 by } \frac{1}{4}
\]

\[
r_2 - 2r_1 \quad \text{means to subtract twice row 1 from row 2.}
\]

Next work on column 2 to put the matrix in the reduced form \[
\begin{bmatrix}
1 & 0 & k_1 \\
0 & 1 & k_2
\end{bmatrix}
\].

\[
r_1 = \frac{2}{15}r_2 \quad \text{means to multiply row 2 by } \frac{2}{15}
\]
Self-Check 7
Write the solution of the system of linear equations represented by:

\[
\begin{pmatrix}
1 & 0 & 2 \\
0 & 1 & 3
\end{pmatrix}
\]

We encourage you to compare this Example 7 to the Example 7 in Section 3.5 where the same problem was worked by the addition method. Example 8 examines a consistent system of dependent equations.
Example 8  Solving a Consistent System of Dependent Equations by Using an Augmented Matrix

Use an augmented matrix and elementary row operations to solve \[
\begin{cases}
-4x + 10y = -2 \\
6x - 15y = 3
\end{cases}
\].

**Solution**

First form the augmented matrix.

\[
\begin{bmatrix}
-4 & 10 & -2 \\
6 & -15 & 3
\end{bmatrix}
\]

Then work on putting column 1 in the form \[
\begin{bmatrix}
1 & -5/2 & 1/2 \\
0 & 0 & 0
\end{bmatrix}
\].

The last row corresponds to an equation that is an identity. It cannot be used to simplify column 2 further.

These two equations form a dependent system of equations with an infinite number of solutions.

Because the coefficient of \(x\) in the first equation is 1, solve this equation for \(x\) in terms of \(y\). You can also solve this equation for \(y\) in terms of \(x\) and express the answer in the form \(\left(\frac{2}{5}y - \frac{1}{5}\right)\).

You may wish to confirm that both of these represent the same set of points. All the solutions are points on the same line, points that can be written in this form.

**Answer:** There are infinitely many solutions, all having the form \(\left(\frac{5}{2}y + \frac{1}{2}, y\right)\).

Self-Check 8

Write the solution of the system of linear equations represented by \[
\begin{bmatrix}
1 & -1 & 4 \\
0 & 0 & 0
\end{bmatrix}
\].

The **general solution** of a system of dependent linear equations describes all solutions of the system and is given by indicating the relationship between the coordinates of the solutions. The **particular solutions** obtained from the general solution contain only constant coordinates.

The general solution in Example 8 is \(\left(\frac{5}{2}y + \frac{1}{2}, y\right)\). By arbitrarily selecting \(y\)-values, we can produce as many particular solutions as we wish. For \(y = 0\) and \(y = 1\), we obtain two particular solutions \(\left(\frac{1}{2}, 0\right)\) and \((3, 1)\).

Self-Check Answers

1. \[
\begin{bmatrix}
5 & 1 & -3 \\
3 & 2 & 8
\end{bmatrix}
\]
2. \[
\begin{cases}
4x - 3y = 1 \\
2x + 9y = 4
\end{cases}
\]
3. \((5, 6)\)
4. a. \[
\begin{bmatrix}
1 & -1 & -1 \\
3 & 2 & 12
\end{bmatrix}
\]
   b. \[
\begin{bmatrix}
1 & 1 & 9 \\
0 & 1 & 5
\end{bmatrix}
\]
   c. \[
\begin{bmatrix}
1 & 3 & 7 \\
0 & 1 & 3
\end{bmatrix}
\]
5. \((-8, 1)\)
6. \((1.5, -1)\)
7. There is no solution; it is an inconsistent system.
8. A dependent system with a general solution \((y + 4, y)\); three particular solutions are \((4, 0), (5, 1),\) and \((6, 2)\).
1. A matrix is a ________ array of numbers.
2. The entries in an augmented matrix for a system of linear equations consist of the ________ and ________ in the equations.
3. ________ systems of equations have the same solution set.
4. The notation $r_1 \leftrightarrow r_2$ denotes that rows 1 and 2 for a matrix are ________.
5. The notation $r_i = \frac{1}{2}r_1$ denotes that row ________ is being replaced by multiplying the current row ________ by ________.

6. The notation to represent that row 2 is being replaced by the sum of the current row 2 plus twice row 1 is ________.

7. The ________ solution of a system of dependent linear equations describes all solutions of the system and is given by indicating the relationship between the coordinates of the solutions.

8. The ________ solutions obtained from a general solution contain only constant coordinates.

12.1 Using the Language and Symbolism of Mathematics

In Exercises 1–6, write an augmented matrix for each system of linear equations.

1. $3x + y = 0$
   $2x - y = -5$

2. $5x - y = 3$
   $4x + 3y = 29$

3. $4x = 12$
   $3y = 3$

4. $2x + 7y = 3$
   $-3y = 3$

5. $x = 5$
   $y = -6$

6. $x = -4$
   $y = 11$

In Exercises 7–12, write a system of linear equations in $x$ and $y$ that is represented by each augmented matrix.

7. $\begin{bmatrix} 2 & 3 & 2 \\ 4 & -3 & 1 \end{bmatrix}$

8. $\begin{bmatrix} 5 & -1 & 0 \\ 3 & 2 & 13 \end{bmatrix}$

9. $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \end{bmatrix}$

10. $\begin{bmatrix} 2 & 3 & 5 \\ 6 & -4 & 2 \end{bmatrix}$

11. $\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & -8 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 0 & \frac{2}{3} \\ 0 & 1 & -\frac{4}{5} \end{bmatrix}$

12.1 Quick Review

1. Determine whether $(-2, 5)$ is a solution of the system
   \[
   \begin{align*}
   4x + 3y &= 7 \\
   5x + 2y &= 1
   \end{align*}
   \]

2. If the $x$-coordinate of the solution of
   \[
   \begin{align*}
   2x + 3y &= 4 \\
   3x + 2y &= 11
   \end{align*}
   \]
   is 5, determine the $y$-coordinate.

3. If both sides of the equation $\frac{x}{2} - \frac{y}{3} = 1$ are multiplied by 6, the resulting equivalent equation is ________.

4. Solve \[
   \begin{align*}
   3x - 4y &= 8 \\
   x - 2y &= 2
   \end{align*}
   \]
   by the substitution method.

5. Solve \[
   \begin{align*}
   2x + 5y &= 1 \\
   3x - 4y &= 13
   \end{align*}
   \]
   by the addition method.
In Exercises 13–20, use the given elementary row operations to complete each matrix.

13. \[
\begin{bmatrix}
2 & 1 & 1 \\
1 & 3 & 0
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 0 \\
2 & 1 & 1
\end{bmatrix}
\]
14. \[
\begin{bmatrix}
-6 & 3 & 4 \\
1 & 2 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 2 & 1 \\
-2 & 1 & 1
\end{bmatrix}
\]
15. \[
\begin{bmatrix}
1 & -3 & -2 \\
2 & -5 & 4
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & -5 & 4 \\
1 & -3 & -2
\end{bmatrix}
\]
16. \[
\begin{bmatrix}
1 & -1 & 1 \\
2 & -3 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
2 & -3 & -2 \\
1 & -1 & 1
\end{bmatrix}
\]
17. \[
\begin{bmatrix}
3 & -1 & 3 \\
6 & 4 & -6
\end{bmatrix}
\rightarrow
\begin{bmatrix}
6 & 4 & -6 \\
3 & -1 & 3
\end{bmatrix}
\]
18. \[
\begin{bmatrix}
2 & 3 & 1 \\
3 & -3 & -21
\end{bmatrix}
\rightarrow
\begin{bmatrix}
3 & -3 & -21 \\
2 & 3 & 1
\end{bmatrix}
\]
19. \[
\begin{bmatrix}
1 & 5 & 16 \\
0 & 1 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 5 & 16 \\
0 & 1 & 3
\end{bmatrix}
\]
20. \[
\begin{bmatrix}
1 & -4 & 10 \\
0 & 1 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -4 & 10 \\
0 & 1 & -2
\end{bmatrix}
\]

In Exercises 21–26, write the solution for the system of linear equations represented by each augmented matrix. If the matrix represents a consistent system of dependent equations, write the general solution and three particular solutions.

21. \[
\begin{bmatrix}
1 & 0 & -5 \\
0 & 1 & 9
\end{bmatrix}
\]
22. \[
\begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 6
\end{bmatrix}
\]
23. \[
\begin{bmatrix}
1 & 4 & 3 \\
0 & 0 & 7
\end{bmatrix}
\]
24. \[
\begin{bmatrix}
1 & -3 & 5 \\
0 & 0 & -2
\end{bmatrix}
\]
25. \[
\begin{bmatrix}
1 & 3 & 5 \\
0 & 0 & 0
\end{bmatrix}
\]
26. \[
\begin{bmatrix}
1 & -2 & 3 \\
0 & 0 & 0
\end{bmatrix}
\]

In Exercises 27–34, label the elementary row operation used to transform the first matrix to the second. Use the notation developed in this section.

27. \[
\begin{bmatrix}
2 & 6 & 8 \\
3 & 7 & 10
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 4 \\
3 & 7 & 10
\end{bmatrix}
\]
28. \[
\begin{bmatrix}
2 & 5 & 9 \\
3 & 2 & 8
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 5 & 9 \\
3 & 2 & 8
\end{bmatrix}
\]
29. \[
\begin{bmatrix}
1 & 5 & 8 \\
3 & 2 & -2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 5 & 8 \\
0 & -13 & -26
\end{bmatrix}
\]
30. \[
\begin{bmatrix}
1 & 3 & 8 \\
2 & -3 & 7
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 3 & 8 \\
0 & -9 & -9
\end{bmatrix}
\]
31. \[
\begin{bmatrix}
1 & -2 & 8 \\
0 & 3 & -9
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & -2 & 8 \\
0 & 1 & -3
\end{bmatrix}
\]
32. \[
\begin{bmatrix}
1 & 6 & 2 \\
0 & 3 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 6 & 2 \\
0 & 1 & 3
\end{bmatrix}
\]
33. \[
\begin{bmatrix}
1 & 2 & 6 \\
0 & 1 & 1
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 4 \\
0 & 1 & 1
\end{bmatrix}
\]

34. \[
\begin{bmatrix}
1 & 3 & 2 \\
0 & 1 & 3
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & 1 \\
0 & 1 & 3
\end{bmatrix}
\]

In Exercises 35–50, use an augmented matrix and elementary row operations to solve each system of linear equations.

35. \[x + 3y = 5 \quad 2x + y = -5\]
36. \[x - 2y = 9 \quad 3x + 4y = 7\]
37. \[x + 3y = 1 \quad 3x + 7y = 7\]
38. \[x + 2y = 7 \quad 4x + 3y = 3\]
39. \[2x + 5y = -4 \quad 4x + 3y = 6\]
40. \[2x - 3y = 17 \quad 4x + y = 13\]
41. \[4x - 9y = 5 \quad 3x + 12y = 10\]
42. \[8x + 3y = -39 \quad 7x - 2y = -11\]
43. \[3x - y = 2 \quad 2x + y = 6\]
44. \[3x + 4y = 0 \quad 2x - 3y = 16\]
45. \[6x + 4y = 11 \quad 10x + 6y = 17\]
46. \[5x - y = 7 \quad 2x + 4y = 5\]
47. \[3x + 4y = 7 \quad 6x + 8y = 10\]
48. \[4x + 3y = 2 \quad 16x + 12y = 7\]
49. \[2x - y = 5 \quad 4x - 2y = 10\]
50. \[3x - 6y = 12 \quad 4x - 8y = 16\]

Connecting Concepts to Applications

In Exercises 51–60, write a system of linear equations using the variables \(x\) and \(y\), and use this system to solve the problem.

51. Numeric Word Problem Find two numbers whose sum is 160 and whose difference is 4.
52. Numeric Word Problem Find two numbers whose sum is 260 if one number is 3 times the other number.
53. Complementary Angles The two angles shown are complementary, and one angle is 32° larger than the other. Determine the number of degrees in each angle.

54. Supplementary Angles The two angles shown are supplementary, and one angle is 74° larger than the other. Determine the number of degrees in each angle.
55. Fixed and Variable Costs A seamstress makes custom costumes for operas. One month the total of fixed and variable costs for making 20 costumes was $3,200. The next month the total of the fixed and variable costs for making 30 costumes was $4,300. Determine the fixed cost and the variable cost per costume.

56. Rates of Two Bicyclists Two bicyclists depart at the same time from a common location, traveling in opposite directions. One averages 5 km/h more than the other. After 2 hours, they are 130 km apart. Determine the speed of each bicyclist.

57. Rate of a River Current A small boat can go 30 km downstream in 1 hour, but only 14 km upstream in 1 hour. Determine the rate of the boat and the rate of the current.

58. Mixture of Two Disinfectants A hospital needs 100 L of a 15% solution of disinfectant. How many liters of a 40% solution and a 5% solution should be mixed to obtain this 15% solution?

59. Mixture of a Fruit Drink A fruit concentrate is 15% water. How many liters of pure water and how many liters of concentrate should be mixed to produce 100 L of mixture that is 83% water?

60. Basketball Scores During one game for the Phoenix Suns, Steve Nash scored 39 points on 17 field goals. How many of these field goals were 2-pointers and how many were 3-pointers?

61. Radius of a Train Wheel On a curve, the radius $r_1$ of the inside train wheel is less than the radius $r_2$ of the outside wheel. This allows the outside wheel to travel a greater distance as the train goes around the curve. On one curve, both radii are measured in inches, and the result is $r_2 - r_1 = 0.25$. The inside wheel covers 111.5 inches per revolution. Determine the radius of the inside wheel and the radius of the outside wheel.

Group discussion questions

62. Challenge Question Solve \( \begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases} \) for \((x, y)\) in terms of \(a_1, a_2, b_1, b_2, c_1, \) and \(c_2\). Assume that \(a_1b_2 - a_2b_1 \neq 0\).

63. Discovery Question a. Extend the augmented matrix notation given for systems of two linear equations with two variables to write an augmented matrix for this system.
\[
\begin{cases}
x + 2y + 3z = 5 \\
2x + y - z = -1 \\
3x + y + z = 4
\end{cases}
\]
b. Write a system of linear equations that is represented by this augmented matrix. Use the variables \(x, y, \) and \(z\).
\[
\begin{bmatrix}
2 & -3 & 3 & 2 \\
4 & 0 & 2 & -1 \\
-2 & 4 & -3 & -2
\end{bmatrix}
\]
c. Write a system of linear equations that is represented by this augmented matrix. Use the variables \(w, x, y, \) and \(z\).
\[
\begin{bmatrix}
1 & -1 & 3 & -4 & 1 \\
-2 & 4 & -1 & 1 & -4 \\
3 & 2 & -4 & 5 & 7 \\
2 & -1 & -1 & 4 & 7
\end{bmatrix}
\]

12.1 Cumulative Review

1. Write the first five terms of the arithmetic sequence defined by \(a_n = 2n + 1\).

2. Write the 50th term of the arithmetic sequence defined by \(a_n = 2n + 1\).

3. Write the first five terms of the geometric sequence defined by \(a_n = 2^n + 1\).

4. Simplify \((12a + 3b)^0 + (12a)^0 + (3b)^0 + 12a^0 + 3b^0\) assuming all bases are nonzero.

5. Simplify \((-1)^4 + 4^{-1} + (1)^{1/4}\).
Section 12.2 Systems of Linear Equations in Three Variables

Objective:

1. Solve a system of three linear equations in three variables.

A first-degree equation in two variables of the form \( Ax + By = C \) is called a linear equation because its graph is a straight line if \( A \) and \( B \) are not both 0. Similarly, a first-degree equation in three variables of the form \( Ax + By + Cz = D \) also is called a linear equation. However, this name is misleading because if \( A, B, \) and \( C \) are not all 0, the graph of \( Ax + By + Cz = D \) is not a line but a plane in three-dimensional space.

The graph of a three-dimensional space on two-dimensional paper is limited in its portrayal of the third dimension. Nonetheless, we can give the viewer a feeling for planes in a three-dimensional space by orienting the \( x \)-, \( y \)-, and \( z \)-axes as shown in the figure. This graph illustrates the plane whose \( x \)-intercept is \((6, 0, 0)\), whose \( y \)-intercept is \((0, 4, 0)\), and whose \( z \)-intercept is \((0, 0, 3)\). Drawing lines to connect these intercepts gives the view of the plane in the region where all coordinates are positive.

1. Solve a System of Three Linear Equations in Three Variables

A system of three linear equations in three variables is referred to as a (three-by-three) system. A solution of a system of equations with the three variables \( x \), \( y \), and \( z \) is an ordered triple \((x, y, z)\) that is a solution of each equation in the system.

Example 1 Determining Whether an Ordered Triple Is a Solution of a System of Linear Equations

Determine whether \((2, -3, 5)\) is a solution of \( \begin{cases} x + y - z = -6 \\ 2x - y + z = 12 \\ 3x - 2y - 2z = 3 \end{cases} \).

**Solution**

<table>
<thead>
<tr>
<th>First Equation</th>
<th>Second Equation</th>
<th>Third Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x + y - z = -6 )</td>
<td>( 2x - y + z = 12 )</td>
<td>( 3x - 2y - 2z = 3 )</td>
</tr>
<tr>
<td>( 2 + (-3) - 5 \neq -6 )</td>
<td>( 2(2) - (-3) + 5 \neq 12 )</td>
<td>( 3(2) - 2(-3) - 2(5) \neq 3 )</td>
</tr>
<tr>
<td>(-6 \neq -6) is true.</td>
<td>( 4 + 3 + 5 \neq 12 )</td>
<td>( 6 + 6 - 10 \neq 3 )</td>
</tr>
</tbody>
</table>

**Answer:** \((2, -3, 5)\) is not a solution of this system. To be a solution of this system, the point must satisfy all three equations.

Self-Check 1

Determine whether \((2, -3.25, 4.75)\) is a solution of the system in Example 1.

The graph of each linear equation \( Ax + By + Cz = D \) is a plane in three-dimensional space unless \( A, B, \) and \( C \) are all 0. A system of three linear equations in three variables can be viewed geometrically as the intersection of a set of three planes. These planes may intersect in one point, no points, or an infinite number of points. The illustrations in the following box show some of the ways we can obtain these solutions. Can you sketch other ways of obtaining these solution sets?
Can I Use Graphs to Solve Systems of Three Linear Equations with Three Variables?  

No, although the figures in the box can give us an intuitive understanding of the possible solutions to these systems it is not practical to actually solve these systems graphically. Thus we rely entirely on algebraic methods.

In this section, we illustrate two methods for solving systems of three linear equations in three variables (systems). Example 2 extends the substitution and addition methods from Sections 3.4 and 3.5 to solve a $3 \times 3$ system. Later in the section, we use augmented matrices, which were introduced in Section 12.1.

Equivalent systems of equations have the same solution set. The general goal of each step of a solution process is to produce an equivalent system that is simpler than the previous step. By eliminating some of the variables, we can reduce a $3 \times 3$ system to a system with only two variables, and then we can eliminate another variable to produce an equation with only one variable. We can then back-substitute to obtain the values of the other two variables. This strategy is outlined in the following box.

### Strategy for Solving a $3 \times 3$ System of Linear Equations

1. Write each equation in the general form $Ax + By + Cz = D$.
2. Select one pair of equations and use the substitution method or the addition method to eliminate one of the variables.
3. Repeat step 2 with another pair of equations. Be sure to eliminate the same variable as in step 2.
4. Eliminate another variable from the pair of equations produced in steps 2 and 3, and solve this $2 \times 2$ system of equations.
5. Back-substitute the values from step 4 into one of the original equations to solve for the third variable.
6. Does this solution check in all three of the original equations?

*If a contradiction is obtained in any of these steps, the system is inconsistent and has no solution. If an identity is obtained in any step, the system is either dependent with infinitely many solutions or inconsistent with no solution.
**Example 2**  
**Solving a 3 × 3 System of Linear Equations**

Solve the system
\[
\begin{align*}
  x + y + z &= 2 \\
  -x + y - 2z &= 1 \\
  x + y - z &= 0 
\end{align*}
\]

**Solution**

**Produce a 2 × 2 System of Equations**

\[
\begin{align*}
  (1) & \quad x + y + z = 2 \\
  (2) & \quad -x + y - 2z = 1 \\
  (3) & \quad x + y - z = 0 
\end{align*}
\]

The 2 × 2 system of equations is
\[
\begin{align*}
  (1) & \quad 2y - z = 3 \\
  (2) & \quad 2y - 3z = 1 
\end{align*}
\]

**Produce an Equation with Only One Variable**

\[
\begin{align*}
  2y - z &= 3 \\
  2y - 3z &= 1 \\
  \text{Subtract the second equation from the first equation to eliminate } y. \\
  2z &= 2 \\
  z &= 1 
\end{align*}
\]

**Back-Substitute**

\[
\begin{align*}
  2y - z &= 3 & (1) & \quad x + y + z = 2 \\
  2y &= 4 & (2) & \quad x + (2) + (1) = 2 \\
  y &= 2 & x &= 3 \\
  y &= 2 & x &= -1 
\end{align*}
\]

The answer as an ordered triple is (-1, 2, 1).

**Check:**  
\[
\begin{align*}
  (1) & \quad x + y + z = 2 \\
  (2) & \quad -x + y - 2z = 1 \\
  (3) & \quad x + y - z = 0 \\
  -1 + 2 + 1 & \neq 2 \\
  -1 + 2 - 1 & \neq 0 \\
  0 & \neq 0 \text{ is true.} \\
  1 & \neq 1 \text{ is true.} \\
  2 & \neq 2 \text{ is true.} 
\end{align*}
\]

**Answer:**  
The solution (-1, 2, 1) checks in all three equations.

**Self-Check 2**

Solve the system
\[
\begin{align*}
  x + y + z &= 6 \\
  -x + y + 2z &= -1 \\
  x + y - z &= 0 
\end{align*}
\]

We can also use augmented matrices to organize and expedite our work in solving 3 × 3 systems of linear equations. To prepare for this, we examine what matrices for 3 × 3 systems of linear equations look like.

In Example 3, the entries in the first column of the matrix are the coefficients of \( x \), the entries in the second column are the coefficients of \( y \), the entries in the third column are the coefficients of \( z \), and the entries in the fourth column are the constants.
Chapter 12  A Preview of College Algebra

Example 3  Writing Augmented Matrices for Systems of Linear Equations

Write an augmented matrix for each system of linear equations.

Solution

(a) \[
\begin{align*}
  x + y - z &= -6 \\
  2x - y + z &= 12 \\
  3x - 2y - 2z &= 3
\end{align*}
\]

\[
\begin{bmatrix}
  1 & 1 & -1 & -6 \\
  2 & -1 & 1 & 12 \\
  3 & -2 & -2 & 3
\end{bmatrix}
\]

The augmented matrix contains the coefficients and the constants from each equation.

(b) \[
\begin{align*}
  2x + y &= 7 \\
  y - z &= 2 \\
  x + z &= 2
\end{align*}
\]

\[
\begin{bmatrix}
  2 & 1 & 0 & 7 \\
  0 & 1 & -1 & 2 \\
  1 & 0 & 1 & 2
\end{bmatrix}
\]

Use zero coefficients as needed for any missing terms.

Self-Check 3

Write the augmented matrix for this system of linear equations.

\[
\begin{align*}
  x + 2y + z &= -2 \\
  2x + 4y - 3z &= 0 \\
  -x + 2y &= -3
\end{align*}
\]

The reduced form for the augmented matrix associated with a consistent system of three independent linear equations with three variables is \[
\begin{bmatrix}
  1 & 0 & 0 & k_1 \\
  0 & 1 & 0 & k_2 \\
  0 & 0 & 1 & k_3
\end{bmatrix}
\]. The solution of this system is \((k_1, k_2, k_3)\). Similar forms can be obtained for dependent or inconsistent systems. The reduced form defined here is also called reduced row echelon form.

Properties of the Reduced Row Echelon Form of a Matrix

1. The first nonzero entry in a row is a 1. All other entries in the column containing the leading 1 are 0s.
2. All nonzero rows are above any rows containing only 0s.
3. The first nonzero entry in a row is to the left of the first nonzero entry in the following row.

Example 4  Using Elementary Row Operations

Use the elementary row operations to transform the matrix \[
\begin{bmatrix}
  5 & 6 & -4 & -8 \\
  2 & 4 & -1 & 1 \\
  1 & 1 & -3 & 0
\end{bmatrix}
\] into the form \[
\begin{bmatrix}
  1 & \_ & \_ & \_ \\
  0 & \_ & \_ & \_ \\
  0 & \_ & \_ & \_
\end{bmatrix}
\].
12.2 Systems of Linear Equations in Three Variables

The last two steps in Example 4 can be combined, as illustrated in Example 5.

**Example 5**  Solving a System of Linear Equations by Using an Augmented Matrix

Solve
\[
\begin{align*}
2a + 3b - 2c &= -8 \\
-a + b + 2c &= 25 \\
4a + 6b - c &= -7
\end{align*}
\]

**Solution**

The first step is to form the augmented matrix.
\[
\begin{bmatrix}
2 & 3 & -2 & -8 \\
1 & -1 & 2 & 25 \\
4 & 6 & -1 & -7
\end{bmatrix}
\]

Transform the first column into the form
\[
\begin{bmatrix}
1 & -1 & 2 & 25 \\
0 & 3 & -2 & -8 \\
0 & 6 & -1 & -7
\end{bmatrix}
\]

Transform the second column into the form
\[
\begin{bmatrix}
1 & 0 & -6/5 & -58/5 \\
0 & 1 & 0 & -107/5 \\
0 & 0 & 1 & 9
\end{bmatrix}
\]
Self-Check 5

Write an ordered triple for the solution of the system of linear equations represented by
\[
\begin{bmatrix}
1 & 0 & 0 & 6 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & 5
\end{bmatrix}
\]

The overall strategy used in Example 5 can be summarized as “work from left to right and produce the leading 1s before you produce the 0s.” This strategy is formalized in the following box.

**Transforming an Augmented Matrix into Reduced Echelon Form**

**Step 1.**
\[
\begin{bmatrix}
1 & \cdots & 1 \\
0 & \cdots & 0 \\
: & \cdots & : \\
0 & \cdots & 0
\end{bmatrix}
\]

Transform the first column into this form by using the elementary row operations to
- Produce a 1 in the top position.
- Use the 1 in row 1 to produce 0s in the other positions of column 1.

**Step 2.**
\[
\begin{bmatrix}
1 & 0 & \cdots & 1 \\
0 & 1 & \cdots & 0 \\
: & : & \cdots & : \\
0 & 0 & \cdots & 0
\end{bmatrix}
\]

Transform the next column, if possible, into this form by using the elementary operations to
- Produce a 1 in the next row.
- Use the 1 in this row to produce 0s in the other positions of this column.

If it is not possible to produce a 1 in the next row, proceed to the next column.

**Step 3.** Repeat step 2 column by column, always producing the 1 in the next row, until you arrive at the reduced form.

For emphasis, remember that your goal is to produce the leading 1s before you produce the 0s. You may use shortcuts in this process whenever they are appropriate. This procedure also works for dependent and inconsistent systems.
Example 6  
Solving an Inconsistent System by Using an Augmented Matrix

\[
\begin{align*}
\begin{cases}
r + 8s + 2t = 20 \\
11s + t = 28 \\
-22s - 2t = -55
\end{cases}
\end{align*}
\]

Solve the system above.

Solution: 

\[
\begin{bmatrix}
1 & 8 & 2 & 20 \\
0 & 11 & 1 & 28 \\
0 & -22 & -2 & -55
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 8 & 2 & 20 \\
0 & 11 & 1 & 28 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Although this matrix is not in reduced form, the last row indicates that the system is inconsistent with no solution. The last row represents the equation \(0r + 0s + 0t = 1\), which is a contradiction.

Answer: There is no solution.

Self-Check 6

Write the solution for the system of linear equations represented by

\[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 2 & 1 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

Example 7 produces the general solution for a consistent system of dependent linear equations.

Example 7  
Solving a Consistent System of Dependent Equations by Using an Augmented Matrix

\[
\begin{align*}
\begin{cases}
x_1 + 4x_2 + 5x_3 = -2 \\
x_2 + 2x_3 = -1 \\
-5x_2 - 10x_3 = 5
\end{cases}
\end{align*}
\]

Solve the system above.

Solution: 

\[
\begin{bmatrix}
1 & 4 & 5 & -2 \\
0 & 1 & 2 & -1 \\
0 & -5 & -10 & 5
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 0 & -3 & 2 \\
0 & 1 & 2 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

The new matrix is in reduced form. A row of 0s in an \(n \times n\) consistent system indicates a dependent system with infinitely many solutions.

This is the system represented by the reduced matrix. The system is dependent, since the last equation is an identity satisfied by all values of \(x_1, x_2,\) and \(x_3\).

Thus

\[
\begin{align*}
x_1 &= 3x_3 + 2 \\
x_2 &= -2x_3 - 1
\end{align*}
\]

Answer: The general solution is \((3x_3 + 2, -2x_3 - 1, x_3)\); three particular solutions are \((8, -5, 2), (2, -1, 0),\) and \((-7, 5, -3)\).
The matrix method is also a convenient method for solving systems of linear equations that do not have the same number of variables as equations.

Let \( A \) be the augmented matrix of a system of equations.

1. If the reduced form of \( A \) has a row of the form \([0 \ 0 \ \ldots \ 0 \ k]\), where \( k \neq 0 \), then the system is inconsistent and has no solution.

2. If the system is consistent and the reduced form of \( A \) has a row of the form \([0 \ 0 \ \ldots \ 0 \ 0]\) (all zeros), then the equations in the system are dependent and the system has infinitely many solutions.

The matrix method is also a convenient method for solving systems of linear equations that do not have the same number of variables as equations.

**Example 8** Solving a System of Equations with More Variables Than Equations

Find a general solution and three particular solutions for the system of equations \[
\begin{align*}
2x - 3y + 17z &= 12, \\
8x + 2y - 2z &= 20.
\end{align*}
\]

**Solution**

Use the elementary row operations to transform the matrix to reduced row echelon form.

\[
\begin{pmatrix}
2 & -3 & 17 & | & 12 \\
8 & 2 & -2 & | & 20
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -\frac{3}{2} & \frac{17}{2} & | & 6 \\
0 & -8 & -2 & | & 10
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & -\frac{3}{2} & \frac{17}{2} & | & 6 \\
0 & 14 & -5 & | & 6
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 0 & 1 & | & 3 \\
0 & 1 & -5 & | & -2
\end{pmatrix}
\]

This is the system represented by the reduced matrix.

The general solution is obtained by solving these equations for \( x \) and \( y \) in terms of \( z \).

The particular solutions were found by letting \( z \) be 0, -1, and 2, respectively.

**Answer:** The general solution is \((3 - z, 5z - 2, z)\);

particular solutions are \((3, -2, 0), (4, -7, -1),\) and \((1, 8, 2)\).
Some graphing calculators have the capability to produce the reduced form of an augmented matrix. On a TI-84 Plus calculator, the \textit{rref} feature produces the row echelon form of a matrix. This is option \textit{B} in the \textit{MATH} menu of \textit{MATRIX} menu. Before you work through Technology Perspective 12.2.1, first store the matrix from Example 5 in your calculator as Matrix 1. Note that the screen size may not display all entries on one view. You may need to use the arrow keys to scroll through all the entries.

**Self-Check 8**
Write the general solution and one particular solution for the $2 \times 3$ system
\[
\begin{align*}
  x + 2y + z &= 4 \\
  x + 3y + 2z &= 5 
\end{align*}
\]

Technology Perspective 12.2.1
Using a Matrix to Solve a System of Linear Equations

Solve the system of equations from Example 5.

**TI-84 Plus Keystrokes**
1. First store the matrix for Example 5 as Matrix 1.
2. Access the matrix \textit{MATH} menu by pressing \textit{2nd} \textit{MATRX} and then use the down arrow key until you reach option \textit{B}. Choose this \textit{rref} feature by pressing \textit{MATRX} \textit{2nd} \textit{MATRX} \textit{rref}.
3. Next select Matrix 1 as the matrix to reduce by pressing \textit{MATRX} \textit{2nd} \textit{MATRX} \textit{1} and close with a right parenthesis.
4. To perform the row reduction, press \textit{ENTER}.

**TI-84 Plus Screens**

```
\[
\begin{bmatrix}
1 & 0 & 0 & 11 \\
0 & 1 & 0 & -8 \\
0 & 0 & 1 & \splitrule
\end{bmatrix}
\]
```

**Answer:** (11, −8, 3)

**Technology Self-Check 1**
Use the \textit{rref} feature to solve Example 2.

The graphical method and tables of values can be used to solve some systems of linear equations with two variables. Example 9 involves a $4 \times 4$ system with four equations and four variables. Graphs and tables are not appropriate tools for these larger systems because it is difficult to graphically depict more than three dimensions. For these larger systems, augmented matrices are often used with calculators or computers.
Example 9  Solving a System of Four Linear Equations by Using an Augmented Matrix

Use an augmented matrix and a calculator to solve

\[
\begin{align*}
   w + x + 2y - z &= 2 \\
   2w + 4x - 6y + z &= -4 \\
   w - 4x + 5y - 3z &= 11 \\
   3w + 3x - y - 2z &= 5
\end{align*}
\]

**Solution**

\[
\begin{bmatrix}
   1 & 1 & 2 & -1 & 2 \\
   2 & 4 & -6 & 1 & -4 \\
   1 & -4 & 5 & -3 & 11 \\
   3 & 3 & -1 & -2 & -5
\end{bmatrix}
\]

First form the augmented matrix. Note that it is a $4 \times 5$ matrix. Enter this as matrix $[A]$ on a calculator.

\[
\begin{bmatrix}
   1 & 1 & 2 & -1 & 2 \\
   2 & 4 & -6 & 1 & -4 \\
   1 & -4 & 5 & -3 & 11 \\
   3 & 3 & -1 & -2 & -5
\end{bmatrix}
\]

Then use the `rref` feature to transform this matrix to reduced row echelon form.

\[
\begin{bmatrix}
   1 & 0 & 0 & 0 & 10 \\
   0 & 1 & 0 & 0 & -4 \\
   0 & 0 & 1 & 0 & 3 \\
   0 & 0 & 0 & 1 & 10
\end{bmatrix}
\]

Obtain the answer from this reduced form.

**Answer:** $(10, -4, 3, 10)$

Does this answer check?

Self-Check 9

Solve

\[
\begin{align*}
   w + x + y + z &= 8 \\
   w - x - y + z &= 2 \\
   w + x - y - z &= -6 \\
   w - x - y - z &= -8
\end{align*}
\]

Self-Check Answers

1. $(2, -3.25, 4.75)$ is a solution of the system.
2. $(5, -2, 3)$
3. \[
\begin{bmatrix}
   1 & 2 & 1 & -2 \\
   2 & 4 & -3 & 0 \\
   -1 & 2 & 0 & -3
\end{bmatrix}
\]
4. \[
\begin{bmatrix}
   1 & 1 & -3 & 0 \\
   0 & 1 & 11 & -8 \\
   0 & 2 & 5 & 1
\end{bmatrix}
\]
5. $(6, 2, 5)$
6. There is no solution.
7. $(5 + z, z + 2, z)$
8. $(2 + z, 1 - z, z)$; $(2, 1, 0)$
9. $(0, 1, 2, 5)$

Technology Self-Check Answer

1. $(-1, 2, 1)$
12.2 Systems of Linear Equations in Three Variables

1. **Using the Language and Symbolism of Mathematics**

1. The graph of \( Ax + By + Cz = D \), if \( A, B, \) and \( C \) are not all 0, is a _______ in three-dimensional space.

2. A _______ of a system of linear equations with three variables is an ordered triple that satisfies each equation in the system.

3. A \( 3 \times 3 \) system of linear equations contains _______ linear equations with _______ variables.

4. The reduced form for an augmented matrix is also called reduced row _______ form.

5. In reduced row echelon form, a row of the form \([0 \ 0 \ldots \ 0 \ k] \), where \( k \neq 0 \), represents an _______ system of linear equations.

### Quick Review

1. Determine whether \((4, -3)\) is a solution of the system \[
\begin{align*}
2x + 3y &= -1 \\
6x + 5y &= 9
\end{align*}
\]

2. Solve \[
\begin{align*}
x + y &= 5 \\
\frac{2}{3} - \frac{y}{3} &= 1
\end{align*}
\]
by the substitution method.

Choose the letter that best describes each system of equations.

3. \[
\begin{align*}
y &= 2x - 1 \\
y &= 2x + 3
\end{align*}
\]
A. A consistent system of independent equations

4. \[
\begin{align*}
y &= 2x - 1 \\
y &= 2x - 1 \\
y &= 2x - 1
\end{align*}
\]
B. A consistent system of dependent equations

5. \[
\begin{align*}
y &= 2x - 1 \\
y &= x + 3
\end{align*}
\]
C. An inconsistent system

### Exercises

#### Objective 1 Solve a System of Three Linear Equations in Three Variables

In Exercises 1–14, solve each system of three linear equations in three variables.

1. \[
\begin{align*}
x + 2y + z &= 11 \\
x - y + 2z &= 1 \\
2x + y + z &= 4
\end{align*}
\]
2. \[
\begin{align*}
x + 2y + 3z &= 7 \\
x - 3y - 2z &= -13 \\
2x + y + 2z &= -5
\end{align*}
\]
3. \[
\begin{align*}
x + y - z &= 1 \\
2x + y + z &= 4 \\
x - y - 2z &= -2
\end{align*}
\]
4. \[
\begin{align*}
x - y + 2z &= 4 \\
x + 2y - z &= 10 \\
x + y + 3z &= -4
\end{align*}
\]

In Exercises 15–18, write an augmented matrix for each system of linear equations.

15. \[
\begin{bmatrix}
x + y + z &= 2 \\
-x + y - 2z &= 1 \\
x + y - z &= 0
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
x + y - 4z &= -17 \\
-x + y + z &= 6 \\
x + 2y - z &= 19
\end{bmatrix}
\]

17. \[
\begin{bmatrix}
x - 2y &= -1 \\
3x - 2y + 4z &= 32 \\
6x - 3y &= 2
\end{bmatrix}
\]

18. \[
\begin{bmatrix}
3y + 2z &= 1 \\
2x - z &= 5
\end{bmatrix}
\]

In Exercises 19–24, write a system of linear equations in \( x, y, \) and \( z \) that is represented by each augmented matrix.

19. \[
\begin{bmatrix}
1 & -2 & 5 & 0 \\
2 & 4 & -3 & 8 \\
3 & 5 & 7 & 11
\end{bmatrix}
\]

20. \[
\begin{bmatrix}
1 & 2 & 3 & 7 \\
2 & -1 & 2 & 5 \\
1 & -3 & -2 & -13
\end{bmatrix}
\]

21. \[
\begin{bmatrix}
1 & 0 & 2 & 5 \\
0 & 1 & 0 & 0 \\
1 & 1 & 4 & 0
\end{bmatrix}
\]

22. \[
\begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 4
\end{bmatrix}
\]

24. \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
7 \\
-5 \\
8 
\end{bmatrix}
\]
23. \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
2 \\
6 \\
-3 
\end{bmatrix}
\]
In Exercises 25–36, use the given elementary row operations to complete each matrix.
25. \[
\begin{bmatrix}
2 & 1 & -2 \\
1 & 2 & 3 \\
3 & 2 & 1 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
26. \[
\begin{bmatrix}
2 & 3 & 4 \\
1 & -1 & 3 \\
3 & -2 & 2 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
27. \[
\begin{bmatrix}
2 & 4 & 8 \\
3 & 5 & 7 \\
4 & 9 & 2 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
28. \[
\begin{bmatrix}
1 & 2 & 5 \\
0 & 5 & 6 \\
0 & 8 & 2 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
29. \[
\begin{bmatrix}
1 & 3 & 5 \\
4 & 8 & 3 \\
2 & 7 & 9 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
30. \[
\begin{bmatrix}
1 & 3 & 5 \\
4 & 8 & 3 \\
2 & 7 & 9 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
31. \[
\begin{bmatrix}
1 & 3 & 5 \\
4 & 8 & 3 \\
2 & 7 & 9 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
32. \[
\begin{bmatrix}
1 & 3 & 5 \\
4 & 8 & 3 \\
2 & 7 & 9 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
33. \[
\begin{bmatrix}
1 & 3 & 5 \\
4 & 8 & 3 \\
2 & 7 & 9 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
34. \[
\begin{bmatrix}
1 & 3 & 5 \\
4 & 8 & 3 \\
2 & 7 & 9 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
\, & \, & \, \\
\, & \, & \, 
\end{bmatrix}
\]
In Exercises 37–42, write the solution for the system of linear equations represented by each augmented matrix. If the matrix represents a dependent system, write the general solution and three particular solutions.
37. \[
\begin{bmatrix}
1 & 0 & 0 & -2 \\
0 & 1 & 0 \\
0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
5 \\
3 \\
1 
\end{bmatrix}
\]
38. \[
\begin{bmatrix}
1 & 0 & 0 & -9 \\
0 & 1 & 3 \\
0 & 0 & 1 
\end{bmatrix}
\begin{bmatrix}
11 \\
8 \\
7 
\end{bmatrix}
\]
39. \[
\begin{bmatrix}
1 & 0 & 7 & 3 \\
0 & 1 & 2 \\
0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
8 \\
4 \\
0 
\end{bmatrix}
\]
40. \[
\begin{bmatrix}
1 & 0 & 3 & 2 \\
0 & 1 & -3 \\
0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
2 \\
4 \\
7 
\end{bmatrix}
\]
41. \[
\begin{bmatrix}
1 & 0 & 3 & 2 \\
0 & 1 & -3 \\
0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
2 \\
4 \\
7 
\end{bmatrix}
\]
42. \[
\begin{bmatrix}
1 & 0 & 3 & 2 \\
0 & 1 & -3 \\
0 & 0 & 0 
\end{bmatrix}
\begin{bmatrix}
2 \\
4 \\
7 
\end{bmatrix}
\]
In Exercises 43–48, label the elementary row operation used to transform the first matrix to the second. Use the notation introduced in Section 12.1.
43. \[
\begin{bmatrix}
2 & 4 & 6 \\
3 & 1 & -1 \\
1 & 2 & -4 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
3 & 1 & -1 \\
1 & 2 & -4 
\end{bmatrix}
\]
44. \[
\begin{bmatrix}
1 & -2 & 5 \\
0 & 3 & -6 \\
0 & 4 & -3 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
1 & -2 & 5 \\
0 & 3 & -6 
\end{bmatrix}
\]
45. \[
\begin{bmatrix}
-3 & 1 & 1 \\
4 & 2 & -1 \\
1 & 3 & -2 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
-3 & 1 & 1 \\
4 & 2 & -1 
\end{bmatrix}
\]
46. \[
\begin{bmatrix}
0 & 10 & -5 \\
4 & 2 & -1 \\
0 & 10 & -5 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
0 & 10 & -5 \\
4 & 2 & -1 
\end{bmatrix}
\]
47. \[
\begin{bmatrix}
1 & -3 & 2 \\
0 & 1 & 5 \\
0 & 1 & 0 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
1 & 0 & 17 \\
0 & 1 & 17 
\end{bmatrix}
\]
48. \[
\begin{bmatrix}
0 & 1 & 5 \\
0 & 1 & 0 \\
0 & 1 & 0 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
\, & \, & \, \\
0 & 1 & 5 \\
0 & 0 & 9 
\end{bmatrix}
\]
In Exercises 49–58, use an augmented matrix and elementary row operations to solve each system of linear equations.
49. \[
\begin{cases}
2x + y = 7 \\
x + y = -2
\end{cases}
\]
50. \[
\begin{cases}
x + y = -2 \\
y - z = 2 \\
x + z = 2 
\end{cases}
\]
51. \[
\begin{cases}
x + 2y + z = 11 \\
x - y + 2z = 7 \\
x + y + z = 4
\end{cases}
\]
52. \[
\begin{cases}
x + y + 2z = 4 \\
x - y + z = 10 \\
x + y + z = 4
\end{cases}
\]
53. \[
\begin{cases}
x + 2x - 2x = -7 \\
x + 3x - 2x = -14 \\
4x + 2x + 10x = -4
\end{cases}
\]
54. \[
\begin{cases}
x_1 - 2x_2 + 7x_3 = 3 \\
x_1 + 2x_2 - 3x_3 = -5 \\
x_1 - 11x_2 + 24x_3 = 11
\end{cases}
\]
55. \[5a + b - 2c = -3\]
\[2a + 4b + c = -3\]
\[-3a + 5b - 6c = -21\]

56. \[6a - 3b + 3c = 3\]
\[3a + 3b - c = 5\]
\[5a + 2b - 2c = 4\]

57. \[r + 2s - 5t = 4\]
\[3r - s + 2t = 3\]
\[r + 9s - 22t = 10\]
\[4r - 2s + t = 2\]
\[2r + 4s - 2t = 0\]

In Exercises 59 and 60, find a general solution and two particular solutions for each system.

59. \[2a - b - 3c = -5\]
\[3a + b + 5c = 4\]
60. \[3a - b + 5c = 0\]
\[2a + 2b - 2c = 0\]

**Connecting Concepts to Applications**

61. **Numeric Word Problem** The sum of three numbers is 108. The largest number is 16 less than the sum of the other two numbers. The sum of the largest and the smallest is twice the other number. Find the three numbers.

62. **Numeric Word Problem** The largest of three numbers is 7 times the second number. The second number is 7 times the smallest number. The sum of the numbers is 285. Find the three numbers.

63. **Dimensions of a Triangle** The perimeter of the triangle is 168 cm. The length of the longest side is twice that of the shortest side. The sum of the lengths of the shortest side and the longest side is 48 cm more than the length of side \(b\). Find the length of each side.

64. **Dimensions of a Triangle** Triangle \(ABC\) has sides \(a\), \(b\), and \(c\) with side \(a\) the longest side and side \(b\) the shortest side. The length of the longest side of the triangle is 12 cm less than the sum of the lengths of the other two sides. The length of the shortest side is 10 cm more than one-half the length of side \(c\). Find the length of each side if the perimeter is 188 cm.

65. **Angles of a Triangle** Triangle \(ABC\) has angles \(A\), \(B\), and \(C\) with angle \(A\) the largest angle and angle \(B\) the smallest angle. Angle \(C\) is twice as large as the smallest angle. The largest angle is 9° larger than the sum of the other two angles. Find the number of degrees in each angle. (*Hint: The sum of the angles of a triangle is 180°.*)

66. **Angles of a Triangle** Triangle \(ABC\) has angles \(A\), \(B\), and \(C\) with angle \(A\) the largest angle and angle \(B\) the smallest angle. The smallest angle of this triangle is 78° less than the largest angle. Angle \(C\) is 3 times as large as the smallest angle. How many degrees are in each angle?

67. **Mixture of Foods** A zookeeper mixes three foods, the contents of which are described in the following table. How many grams of each food are needed to produce a mixture with 133 g of fat, 494 g of protein, and 1,700 g of carbohydrates?

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fat (%)</td>
<td>6</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>Protein (%)</td>
<td>15</td>
<td>18</td>
<td>20</td>
</tr>
<tr>
<td>Carbohydrates (%)</td>
<td>45</td>
<td>65</td>
<td>70</td>
</tr>
</tbody>
</table>

68. **Use of Farmland** A farmer must decide how many acres of each of three crops to plant during this growing season. The farmer must pay a certain amount for seed and devote a certain amount of labor and water to each acre of crop planted, as shown in the following table.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Seed cost ($)</td>
<td>120</td>
<td>85</td>
<td>80</td>
</tr>
<tr>
<td>Hours of labor</td>
<td>4</td>
<td>12</td>
<td>8</td>
</tr>
<tr>
<td>Gallons of water</td>
<td>500</td>
<td>900</td>
<td>700</td>
</tr>
</tbody>
</table>

The amount of money available to pay for seed is $26,350. The farmer’s family can devote 2,520 hours to tending the crops, and the farmer has access to 210,000 gal of water for irrigation. How many acres of each crop would use all these resources?

**Connecting Algebra to Geometry**

In Exercises 69 and 70, match each graph with the linear equation that defines this plane.

69. \[2x + y + 3z = 6\]

70. \[3x + 2y + 2z = 6\]
In Exercises 71 and 72, graph the plane defined by each linear equation.

71. \(4x + 2y + 3z = 12\)  
72. \(2x + 3y + 6z = 6\)

**Group discussion questions**

73. **Discovery Question**
   a. Solve \(\begin{cases} 3a - 5b = 1 \\ a + 2b = 4 \end{cases}\).
   b. Use the solution for part a to solve this nonlinear system:
      \[
      \begin{aligned}
      \frac{3}{x} - \frac{5}{y} &= 1 \\
      \frac{1}{x} + \frac{2}{y} &= 4
      \end{aligned}
      \]

74. **Discovery Question**
   a. Solve \(\begin{cases} a - 2b - c = 5 \\ 2a - b + 3c = 11 \\ -3a - 2b + 2c = 7 \end{cases}\).
   b. Use the solution for part a to solve this nonlinear system:
      \[
      \begin{aligned}
      \frac{1}{x} - \frac{2}{y} - \frac{1}{z} &= 5 \\
      \frac{2}{x} - \frac{1}{y} + \frac{3}{z} &= 11 \\
      -\frac{3}{x} + \frac{2}{y} + \frac{2}{z} &= 7
      \end{aligned}
      \]

75. **Challenge Question** Find the constants \(a, b,\) and \(c\) such that \((1, -3, 5)\) is a solution of the linear system
   \[
   \begin{cases}
   ax + by + cz = 5 \\
   ax - by - cz = -1 \\
   2ax + 3by + 4cz = 13
   \end{cases}
   \]

76. **Challenge Question** Solve the following system for \((x, y, z)\) in terms of the nonzero constants \(a, b,\) and \(c.\)
   \[
   \begin{cases}
   ax + by + cz = 0 \\
   2ax - by + cz = 14 \\
   -ax + by + 2cz = -21
   \end{cases}
   \]

**Section 12.3**  
**Horizontal and Vertical Translations of the Graphs of Functions**

**Objective:**

1. Analyze and use horizontal and vertical translations.

1. **Analyze and Use Horizontal and Vertical Translations**

Each type of algebraic equation \(y = f(x)\) generates a specific family of graphs. Some of the families that we have examined in earlier chapters are linear functions, absolute value functions, quadratic functions, rational functions, square root functions, exponential functions, and logarithmic functions. Each member of a family of functions shares a characteristic shape with all other members of this family. Now we will use one basic graph from a family to generate other graphs in this family.
What Is Meant by a Vertical Translation of a Graph?

This is a shift up or down of the graph. To analyze these shifts we start by observing the patterns illustrated by the following functions. Observing patterns is an important part of mathematics. By recognizing patterns, we can gain insights that deepen our understanding and make us more productive. The functions compared next are \( y_1 = |x| \) and \( y_2 = |x| + 2 \).

| x   | \( y_1 = |x| \) | \( y_2 = |x| + 2 \) |
|-----|----------------|---------------------|
| -3  | 3              | 5                   |
| -2  | 2              | 4                   |
| -1  | 1              | 3                   |
| 0   | 0              | 2                   |
| 1   | 1              | 3                   |
| 2   | 2              | 4                   |
| 3   | 3              | 5                   |

**Verbally**

Each \( y_2 \)-value is 2 more than the corresponding \( y_1 \)-value. The graph of \( y_2 \) can be obtained by shifting the graph of \( y_1 \) up 2 units.

The graph of \( y_2 = |x| + 2 \) can be obtained by vertically shifting or translating the graph of \( y_1 = |x| \) up 2 units. **Vertical translations** or **vertical shifts** are described in the following box.

### Vertical Shifts of \( y = f(x) \)

For a function \( y_1 = f(x) \) and a positive real number \( c \):

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Graphically</th>
<th>Numerically</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_2 = f(x) + c )</td>
<td>To obtain the graph of ( y_2 = f(x) + c ), shift the graph of ( y_1 = f(x) ) up ( c ) units.</td>
<td>For the same ( x )-value, ( y_2 = y_1 + c ).</td>
</tr>
<tr>
<td>( y_3 = f(x) - c )</td>
<td>To obtain the graph of ( y_3 = f(x) - c ), shift the graph of ( y_1 = f(x) ) down ( c ) units.</td>
<td>For the same ( x )-value, ( y_3 = y_1 - c ).</td>
</tr>
</tbody>
</table>

The graph of each quadratic equation function \( f(x) = ax^2 + bx + c \) is a parabola (see Section 8.4). In Example 1, note that the vertex of \( y_1 = x^2 \) is \((0, 0)\) and the vertex of \( y_2 = x^2 - 4 \) is \((0, -4)\).
Example 1  Examining a Vertical Shift of a Quadratic Function

Compare the functions $y_1 = x^2$ and $y_2 = x^2 - 4$ numerically, graphically, and verbally.

Solution

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 = x^2$</th>
<th>$y_2 = x^2 - 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>$-2$</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>$-1$</td>
<td>1</td>
<td>$-3$</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>$-4$</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$-3$</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>5</td>
</tr>
</tbody>
</table>

 Verbally

For each value of $x$, $y_2$ is 4 units less than $y_1$.

Graphically

The parabola defined by $y_2 = x^2 - 4$ can be obtained by shifting the parabola defined by $y_1 = x^2$ down 4 units. The range of $y_1$ is $[0, \infty)$. The range of $y_2$ is $[-4, \infty)$.

Self-Check 1

Compare the graphs of $y_1 = x^2$ and $y_2 = x^2 + 3$.

Examples 2 and 3 illustrate how to recognize functions that are vertical shifts of each other. Example 2 examines two graphs, and Example 3 examines two tables of values.

Example 2  Identifying a Vertical Shift

Use the graphs of $y_1 = f_1(x)$ and $y_2 = f_2(x)$ to write an equation for $y_2$ in terms of $f_1(x)$.

Solution

The graph of $y_2 = f_2(x)$ can be obtained by shifting the graph of $y_1 = f_1(x)$ up 2 units. Thus $y_2 = f_1(x) + 2$.

Key points shifted are as follows:

- $(0, 1)$ is shifted up 2 units to $(0, 3)$.
- $(1, 2)$ is shifted up 2 units to $(1, 4)$.
- $(3, 1)$ is shifted up 2 units to $(3, 3)$.
- $(4, 1)$ is shifted up 2 units to $(4, 3)$.

Answer: $y_2 = f_1(x) + 2$
12.3 Horizontal and Vertical Translations of the Graphs of Functions

Self-Check 2
Use the graph of \( y_1 = f(x) \) to graph \( y_2 = f(x) + 2 \).

Example 3 Identifying a Vertical Shift
Use the table of values for \( y_1 = f_1(x) \) and \( y_2 = f_2(x) \) to write an equation for \( y_2 \) in terms of \( f_1(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 = f_1(x) )</th>
<th>( y_2 = f_2(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>18</td>
<td>13</td>
</tr>
<tr>
<td>-2</td>
<td>6</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>-5</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>-5</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>-7</td>
</tr>
<tr>
<td>3</td>
<td>-9</td>
<td>-14</td>
</tr>
</tbody>
</table>

Solution
For each value of \( x \), \( y_2 \) is 5 units less than \( y_1 \). Thus \( y_2 = f_1(x) - 5 \).

Answer: \( y_2 = f_1(x) - 5 \)

Self-Check 3
Use the table for \( y_1 = f(x) \) to complete the table for \( y_2 = f(x) + 1 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>-8</td>
<td></td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>8</td>
<td></td>
</tr>
</tbody>
</table>

Since Vertical Shifts Affect the \( y \)-values of a Function, Do Horizontal Shifts Affect the \( x \)-values?
Yes, we will start our inspection of horizontal shifts by examining the functions \( y_1 = |x| \) and \( y_2 = |x + 2| \).

Numerically

| \( x \) | \( y_1 = |x| \) | \( x \) | \( y_2 = |x + 2| \) |
|-------|----------------|-------|----------------|
| -3    | 3              | -5    | 3              |
| -2    | 2              | -4    | 2              |
| -1    | 1              | -3    | 1              |
| 0     | 0              | -2    | 0              |
| 1     | 1              | -1    | 1              |
| 2     | 2              | 0     | 2              |
| 3     | 3              | 1     | 3              |

Graphically
Verbally
For the same y-value in \( y_1 \) and \( y_2 \), the x-value is 2 units less for \( y_2 \) than for \( y_1 \). The graph of \( y_2 \) can be obtained by shifting the graph of \( y_1 \) to the left 2 units.

**Horizontal Shifts of \( y = f(x) \)**

For a function \( y_1 = f(x) \) and a positive real number \( c \):

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Graphically</th>
<th>Numerically</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y_2 = f(x + c) )</td>
<td>To obtain the graph of ( y_2 = f(x + c) ), shift the graph of ( y_1 = f(x) ) to the left ( c ) units.</td>
<td>For the same y-value in ( y_1 ) and ( y_2 ), the x-value is ( c ) units less for ( y_2 ) than for ( y_1 ).</td>
</tr>
<tr>
<td>( y_3 = f(x - c) )</td>
<td>To obtain the graph of ( y_3 = f(x - c) ), shift the graph of ( y_1 = f(x) ) to the right ( c ) units.</td>
<td>For the same y-value in ( y_1 ) and ( y_2 ), the x-value is ( c ) units more for ( y_2 ) than for ( y_1 ).</td>
</tr>
</tbody>
</table>

**Example 4** Examining a Horizontal Shift of a Quadratic Function

Compare the functions \( f_1(x) = x^2 \) and \( f_2(x) = (x - 2)^2 \) numerically, graphically, and verbally.

**Solution**

**Numerically**

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f_1(x) = x^2 )</th>
<th>( x )</th>
<th>( f_2(x) = (x - 2)^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>

**Verbally**

For the same y-value in \( f_1 \) and \( f_2 \), the x-value is 2 units more for \( f_2 \) than for \( f_1 \). The graph of \( f_2(x) = (x - 2)^2 \) can be obtained by shifting the parabola defined by \( f_1(x) = x^2 \) to the right 2 units. The vertex of \( f_1(x) = x^2 \) is \((0, 0)\). The vertex of \( f_2(x) = (x - 2)^2 \) is \((2, 0)\).

**Self-Check 4**

Compare the graphs of \( f(x) = x^2 \) and \( f(x) = (x + 1)^2 \).

Example 5 examines two graphs that are horizontal translations or shifts of each other.
12.3 Horizontal and Vertical Translations of the Graphs of Functions

**Example 5** Identifying a Horizontal Shift

Use the graphs of \( y_1 = f(x) \) and \( y_2 = f_2(x) \) to write an equation for \( y_2 \) in terms of \( f_1(x) \).

**Solution**

Note that the graph of \( y_2 = f_2(x) \) can be obtained by shifting the graph of \( y_1 = f_1(x) \) to the right 5 units. Thus \( y_2 = f_1(x - 5) \).

Key points shifted are as follows:
- \((-4, 1)\) is shifted right 5 units to \((1, 1)\).
- \((-3, 2)\) is shifted right 5 units to \((2, 2)\).
- \((0, 3)\) is shifted right 5 units to \((5, 3)\).

**Answer:** \( y_2 = f_1(x - 5) \)

**Self-Check 5**

Use the graph of \( y_1 = f(x) \) to graph \( y_2 = f(x - 3) \).

Example 6 examines horizontal shifts when two functions are defined by a table of values.

**Example 6** Using Translations to Report Sales Data

When working with data for which the input values are years, analysts often make use of a horizontal translation to work with smaller input values but with equivalent results. The following tables represent the sales figures for a company founded in 2005. Use the information in the tables to write an equation for \( y_2 = f_2(x) \) as a translation of \( y_1 = f_1(x) \).

<table>
<thead>
<tr>
<th>Year ( x )</th>
<th>Sales ( y_1 ) ($)</th>
<th>Year ( x )</th>
<th>Sales ( y_2 ) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2005</td>
<td>100,000</td>
<td>0</td>
<td>100,000</td>
</tr>
<tr>
<td>2006</td>
<td>120,000</td>
<td>1</td>
<td>120,000</td>
</tr>
<tr>
<td>2007</td>
<td>150,000</td>
<td>2</td>
<td>150,000</td>
</tr>
<tr>
<td>2008</td>
<td>200,000</td>
<td>3</td>
<td>200,000</td>
</tr>
<tr>
<td>2009</td>
<td>270,000</td>
<td>4</td>
<td>270,000</td>
</tr>
<tr>
<td>2010</td>
<td>350,000</td>
<td>5</td>
<td>350,000</td>
</tr>
</tbody>
</table>

**Solution**

Note that 2005 corresponds to year 0, 2006 to year 1, and so on. We can obtain a table of values for \( y_2 \) based on the table for \( y_1 \) by shifting the input \( x \)-values to the left 2005 units. Thus \( y_2 = f_1(x + 2005) \).

**Answer:** \( y_2 = f_1(x + 2005) \)

**Self-Check 6**

The following two tables show the number of students who completed a graphics design program that started in 1990 at a Midwestern community college.

<table>
<thead>
<tr>
<th>Year ( x )</th>
<th>Students ( y_1 )</th>
<th>Year ( x )</th>
<th>Students ( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1995</td>
<td>2</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>2000</td>
<td>6</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>2005</td>
<td>14</td>
<td>15</td>
<td>14</td>
</tr>
<tr>
<td>2010</td>
<td>20</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

Use the information in the tables to write an equation for \( y_2 = f_2(x) \) as a translation of \( y_1 = f_1(x) \).
Example 7 gives some additional practice recognizing horizontal and vertical shifts from the equation defining the function.

**Example 7**  Identifying Horizontal and Vertical Shifts

Describe how to shift the graph of \( y = f(x) \) to obtain the graph of each function.

**Solution**

(a) \( y = f(x) + 8 \)  
Shift the graph of \( y = f(x) \) up 8 units.

(b) \( y = f(x) - 8 \)  
Shift the graph of \( y = f(x) \) down 8 units.

(c) \( y = f(x + 8) \)  
Shift the graph of \( y = f(x) \) left 8 units.

(d) \( y = f(x - 8) \)  
Shift the graph of \( y = f(x) \) right 8 units.

**Self-Check 7**

Describe the shift in the graph of \( y = f(x) \) to obtain the graph of each function.

a. \( y = f(x - 2) \)  
b. \( y = f(x) - 2 \)  
c. \( y = f(x) + 2 \)  
d. \( y = f(x + 2) \)

**Can I Combine Horizontal and Vertical Shifts in the Same Problem?**

Yes, Example 8 illustrates how to do this.

**Example 8**  Combining Horizontal and Vertical Shifts

Use the graphs of \( y_1 = f_1(x) \) and \( y_2 = f_2(x) \) to write an equation for \( y_2 \) in terms of \( f_1(x) \). Give the domain and range of each function.

**Solution**

The graph of \( y_2 = f_2(x) \) can be obtained by shifting the graph of \( y_1 = f_1(x) \) to the right 3 units and up 1 unit. Note how this shift affects the domain and range.

Key points shifted are as follows:

(0, 0) is shifted right 3 units and up 1 unit to (3, 1).

(2, 3) is shifted right 3 units and up 1 unit to (5, 4).

**Answer:**

\[ y_2 = f_1(x - 3) + 1 \]

Domain of \( y_1 \): \( D = [0, 2] \)

Range of \( y_1 \): \( R = [0, 3] \)

Domain of \( y_2 \): \( D = [3, 5] \)

Range of \( y_2 \): \( R = [1, 4] \)

**Self-Check 8**

Use the graph of \( y_1 = f_1(x) \) and \( y_2 = f_2(x) \) to write an equation for \( y_2 \) in terms of \( f_1(x) \).
The parabola \( f(x) = x^2 \) has a vertex of \((0, 0)\). This parabola can be shifted horizontally and vertically to produce many other parabolas. The vertex is a key point on each of these parabolas. Example 9 gives practice in identifying the vertex of a parabola from the defining equation.

**Example 9**  
**Determining the Vertex of a Parabola**

Determine the vertex of each parabola by using the fact that the vertex of \( f(x) = x^2 \) is \((0, 0)\).

**Solution**

(a) \( f(x) = x^2 + 7 \)  
Vertex of \((0, 7)\)  
This is a vertical shift 7 units up of every point on \( f(x) = x^2 \).

(b) \( f(x) = (x + 7)^2 \)  
Vertex of \((-7, 0)\)  
This is a horizontal shift 7 units left of every point on \( f(x) = x^2 \).

(c) \( f(x) = (x - 7)^2 + 9 \)  
Vertex of \((7, 9)\)  
This is a horizontal shift 7 units right and a vertical shift 9 units up of every point on \( f(x) = x^2 \).

(d) \( f(x) = (x + 8)^2 - 6 \)  
Vertex of \((-8, -6)\)  
This is a horizontal shift 8 units left and a vertical shift 6 units down of every point on \( f(x) = x^2 \).

**Self-Check 9**

Determine by inspection the vertex of the parabola defined by each function.

a. \( f(x) = (x - 5)^2 \)  
b. \( f(x) = x^2 - 5 \)  
c. \( f(x) = (x + 6)^2 - 2 \)  
d. \( f(x) = (x - 4)^2 + 3 \)

In Example 9(c), you can think of the shifts following the same order as the order of operations in the expression. First we subtract inside the parentheses, producing a shift 7 units to the right. Later we add 9, producing a shift 9 units up.

**Self-Check Answers**

1. For each value of \( x \), \( y_2 \) is 3 units more than \( y_1 \). The parabola defined by \( y_2 = x^2 + 3 \) can be obtained by shifting the parabola defined by \( y_1 = x^2 \) up 3 units.

2. The parabola \( f(x) = (x + 1)^2 \) can be obtained by shifting the parabola defined by \( f(x) = x^2 \) to the left 1 unit.

3. \( x \) \( y_1 \) \( y_2 \)  
\[ \begin{array}{c|c|c} 
-2 & -8 & -7 \\
-1 & -1 & 0 \\
0 & 0 & 1 \\
1 & 1 & 2 \\
2 & 8 & 9 \\
\end{array} \]

5.  
6. \( y_2 = f_1(x + 1,990) \)

7. a. Shift the graph of \( y = f(x) \) to the right 2 units.
   b. Shift the graph of \( y = f(x) \) down 2 units.
   c. Shift the graph of \( y = f(x) \) up 2 units.
   d. Shift the graph of \( y = f(x) \) to the left 2 units.

8. \( y_2 = f_1(x - 2) + 4 \)

9. a. \((5, 0)\)  
b. \((0, -5)\)  
c. \((-6, -2)\)  
d. \((4, 3)\)
12.3 Using the Language and Symbolism of Mathematics

1. Each member of a family of functions has a graph with the same characteristic _________.
2. A shift of a graph up or down is called a _________ shift or a _________ translation.
3. A shift of a graph left or right is called a _________ shift or a _________ translation.
4. For a function \( y = f(x) \) and a positive real number \( c \):
   a. \( y = f(x) + c \) will produce a _________ translation \( c \) units _________.
   b. \( y = f(x) - c \) will produce a _________ translation \( c \) units _________.
   c. \( y = f(x + c) \) will produce a _________ translation \( c \) units _________.
   d. \( y = f(x - c) \) will produce a _________ translation \( c \) units _________.
5. For a function \( y = f(x) \) and positive real numbers \( h \) and \( k \):
   a. \( y = f(x + h) + k \) will produce a _________ translation \( h \) units _________ and a _________ translation \( k \) units _________.
   b. \( y = f(x - h) - k \) will produce a _________ translation \( h \) units _________ and a _________ translation \( k \) units _________.
6. The shape of the graph of \( f(x) = x^2 \) is called a _________, and the lowest point on this graph is called its _________.

12.3 Quick Review

Make a quick sketch of the graph of each function.

1. \( f(x) = x \)
2. \( f(x) = x^2 \)
3. \( f(x) = |x| \)
4. \( f(x) = \sqrt{x} \)
5. \( f(x) = 2^x \)

12.3 Exercises

Objective 1 Analyze and Use Horizontal and Vertical Translations

Exercises 1–4 give functions that are translations of \( f(x) = |x| \). Match each graph to the correct function.

1. \( f(x) = |x| + 3 \)
2. \( f(x) = |x + 3| \)
3. \( f(x) = |x| - 3 \)
4. \( f(x) = |x - 3| \)

A. B. C. D.

Exercises 5–8 give functions that are translations of \( f(x) = x^2 \). Match each graph to the correct function.

5. \( f(x) = x^2 - 5 \)
6. \( f(x) = (x - 5)^2 \)
7. \( f(x) = (x + 5)^2 \)
8. \( f(x) = x^2 + 5 \)

A. B. C. D.
Exercises 9–12 give functions that are translations of $f(x) = \sqrt{x}$. Match each graph to the correct function.

9. $f(x) = \sqrt{x} - 4$
10. $f(x) = \sqrt{x} + 4$
11. $f(x) = \sqrt{x} + 4$
12. $f(x) = \sqrt{x} - 4$

A.

Exercises 13–16 give functions that are translations of $f(x) = |x|$. Match each graph to the correct function.

13. $f(x) = |x - 1| + 2$
14. $f(x) = |x + 1| - 2$
15. $f(x) = |x + 1| + 2$
16. $f(x) = |x - 1| - 2$

A.

In Exercises 17–22, use the given graph of $y = f(x)$ and horizontal and vertical shifts to graph each function. (Hint: First translate the three points labeled on the graph.)

17. $y = f(x) - 3$
18. $y = f(x) + 2$
19. $y = f(x + 4)$
20. $y = f(x - 1)$
21. $y = f(x - 1) + 2$
22. $y = f(x + 1) - 2$

In Exercises 23 and 24, each graph is a translation of the graph of $y = \frac{x}{2}$. Write the equation of each graph. Identify the $y$-intercept of each graph.

23.

24.

In Exercises 25 and 26, use the given table of values for $y = f(x)$ to complete each table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-3</td>
</tr>
<tr>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = f(x) + 10$</th>
<th>$x$</th>
<th>$y = f(x) - 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-3</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>-2</td>
<td>-2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
</tbody>
</table>

In Exercises 27 and 28, use the given tables to write an equation for $y_2$ and $y_3$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 = f(x)$</th>
<th>$x$</th>
<th>$y_2$</th>
<th>$x$</th>
<th>$y_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>4</td>
<td>-3</td>
<td>-3</td>
<td>-3</td>
<td>11</td>
</tr>
<tr>
<td>-2</td>
<td>3</td>
<td>-2</td>
<td>-4</td>
<td>-2</td>
<td>10</td>
</tr>
<tr>
<td>-1</td>
<td>8</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>15</td>
</tr>
<tr>
<td>0</td>
<td>7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>14</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
<td>-7</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>-8</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

27. Write an equation for $y_3$ in terms of $f(x)$.
28. Write an equation for $y_3$ in terms of $f(x)$.
29. Use the given table for $f(x) = 2x - 3$ to complete the table for $y_2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 = f(x)$</th>
<th>$x$</th>
<th>$y_2 = f(x - 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-3</td>
<td>4</td>
<td>-3</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>7</td>
<td>9</td>
<td>7</td>
</tr>
</tbody>
</table>
30. Use the given table for \( f(x) = -|x| \) to complete the table for \( y_2 \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 = f(x) )</th>
<th>( x )</th>
<th>( y_2 = f(x + 1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>-3</td>
<td>-4</td>
<td>-3</td>
</tr>
<tr>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
<td>2</td>
<td>-3</td>
</tr>
</tbody>
</table>

In Exercises 31 and 32, use the given tables to write an equation for \( y_2 \) and \( y_3 \).

31. Write an equation for \( y_2 \) in terms of \( f(x) \).

32. Write an equation for \( y_3 \) in terms of \( f(x) \).

In Exercises 33–38, determine the vertex of each parabola by using the fact that the vertex of \( f(x) = x^2 \) is at \((0, 0)\).

33. \( f(x) = (x - 6)^2 \)
34. \( f(x) = (x + 6)^2 \)
35. \( f(x) = x^2 + 11 \)
36. \( f(x) = x^2 - 11 \)
37. \( f(x) = (x + 5)^2 - 8 \)
38. \( f(x) = (x - 8)^2 + 5 \)

In Exercises 39–44, determine the vertex of the graph of each absolute value function by using the fact that the vertex of \( f(x) = |x| \) is at \((0, 0)\).

39. \( f(x) = |x| - 13 \)
40. \( f(x) = |x| + 14 \)
41. \( f(x) = |x + 15| \)
42. \( f(x) = |x - 10| \)
43. \( f(x) = |x - 7| - 6 \)
44. \( f(x) = |x + 9| + 8 \)

Exercises 45–50 describe a translation of the graph \( y = f(x) \). Match each description to the correct function.

45. A translation eleven units left
   - A. \( y = f(x - 11) + 11 \)
   - B. \( y = f(x + 11) - 11 \)
46. A translation eleven units right
   - C. \( y = f(x - 11) \)
   - D. \( y = f(x + 11) \)
47. A translation eleven units down
   - E. \( y = f(x) - 11 \)
   - F. \( y = f(x) + 11 \)
48. A translation eleven units up
   - G. \( y = f(x) - 11 \)
   - H. \( y = f(x) + 11 \)
49. A translation eleven units right and eleven units up
   - I. \( y = f(x + 11) \)
   - J. \( y = f(x + 11) + 11 \)
50. A translation eleven units left and eleven units down

In Exercises 51–56, determine the domain and range of each function \( f_2 \) given that the domain of a function \( f_1 \) is \( D = [0, 5] \) and the range is \( R = [2, 4] \).

51. The graph of \( f_2 \) is obtained from the graph of \( f_1 \) by shifting it 3 units right.
52. The graph of \( f_2 \) is obtained from the graph of \( f_1 \) by shifting it 5 units left.

53. The graph of \( f_2 \) is obtained from the graph of \( f_1 \) by shifting it 7 units down.
54. The graph of \( f_2 \) is obtained from the graph of \( f_1 \) by shifting it 8 units up.
55. The graph of \( f_2 \) is obtained from the graph of \( f_1 \) by shifting it 4 units left and 5 units up.
56. The graph of \( f_2 \) is obtained from the graph of \( f_1 \) by shifting it 6 units right and 4 units down.

In Exercises 57–64, match each function with its graph. Use the shape of the graph of each function and your knowledge of translations to make your choices.

57. \( f(x) = x - 6 \)
58. \( f(x) = x + 4 \)
59. \( f(x) = |x + 5| \)
60. \( f(x) = |x - 3| \)
61. \( f(x) = \sqrt{x} + 3 \)
62. \( f(x) = (x + 4)^3 \)
63. \( f(x) = \sqrt{x} - 5 \)

64. \( f(x) = (x - 3)^2 \)

65. Write the first five terms of each sequence. (Hint: See Example 6 in Section 1.7.)
   - a. \( a_n = 2n \)
   - b. \( a_n = 2n + 3 \)
   - c. \( a_n = 2(n + 3) \)

66. Write the first five terms of each sequence.
   - a. \( a_n = n^2 \)
   - b. \( a_n = n^2 - 9 \)
67. Shifting the Temperature Scale  Many properties of nature are related to the temperature of the object being studied. The Celsius temperature scale (°C) is the same incrementally as the Kelvin temperature scale (K) except that it is 273° lower (water freezes at 0°C and 273°K). The equation \( V(t) = 4t \) gives the volume in milliliters of a gas as a function of the temperature in degrees Celsius. Write an equation for the volume as a function of the temperature in kelvins and complete the following table.

<table>
<thead>
<tr>
<th>°C</th>
<th>V(°C)</th>
<th>°K</th>
<th>V(°K)</th>
</tr>
</thead>
<tbody>
<tr>
<td>200</td>
<td>800</td>
<td>273</td>
<td>1</td>
</tr>
<tr>
<td>150</td>
<td>600</td>
<td></td>
<td>0</td>
</tr>
<tr>
<td>100</td>
<td>400</td>
<td>273</td>
<td>1</td>
</tr>
<tr>
<td>50</td>
<td>200</td>
<td>273</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

68. Shifting a Time Reference  A company has a function \( R(t) = t^2 \), where \( t \) is the number of years the company has been in business. Convert this formula to give the revenue in terms of the calendar year.

Company was founded in January 1995. Thus 1995 is taken as year 0 for the company, 1996 as year 1, and so on. A formula for its revenue per year in millions of dollars is given by \( R(t) = t^2 \), where \( t \) is the number of years the company has been in business. Convert this formula to give the revenue in terms of the calendar year.

<table>
<thead>
<tr>
<th>Year</th>
<th>Revenue</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
</tr>
<tr>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>5</td>
<td>25</td>
</tr>
</tbody>
</table>

69. Shifting a Pricing Function  The price in dollars of five different types of networking cables is given in the table. Because of increased overhead cost, the price of each item is increased by $1. Form a table that gives the new price of each item.

<table>
<thead>
<tr>
<th>Item No.</th>
<th>Price P(x) ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>24.95</td>
</tr>
<tr>
<td>2</td>
<td>33.79</td>
</tr>
<tr>
<td>3</td>
<td>12.98</td>
</tr>
<tr>
<td>4</td>
<td>26.78</td>
</tr>
<tr>
<td>5</td>
<td>19.95</td>
</tr>
</tbody>
</table>

70. Shifting the Distance for Golf

<table>
<thead>
<tr>
<th>Clubs</th>
<th>Distance for Old Clubs (yd)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Iron</td>
<td>200</td>
</tr>
<tr>
<td>Iron</td>
<td>190</td>
</tr>
<tr>
<td>Iron</td>
<td>180</td>
</tr>
<tr>
<td>Iron</td>
<td>165</td>
</tr>
<tr>
<td>Iron</td>
<td>150</td>
</tr>
</tbody>
</table>

Group discussion questions

71. Calculator Discovery  Use a graphing calculator to graph each pair of functions on the same graphing screen. Then describe the relationship between each pair of functions.

a. \( Y_1 = x \)  \( Y_2 = -x \)  b. \( Y_1 = |x| \)  \( Y_2 = -|x| \)  c. \( Y_1 = x^2 \)  \( Y_2 = -x^2 \)  d. \( Y_1 = \sqrt{x} \)  \( Y_2 = -\sqrt{x} \)

72. Calculator Discovery  Use a graphing calculator to graph the functions: \( f(x) = |x|, f(x) = 2|x|, f(x) = 3|x|, \) and \( f(x) = 4|x| \). Can you describe the relationship between the graph of \( f(x) = |x| \) and \( f(x) = c|x| \) for any positive constant \( c \)?

73. Calculator Discovery  Use a graphing calculator to graph the functions: \( f(x) = -x^2, f(x) = -2x^2, f(x) = -3x^2, f(x) = -4x^2 \). Can you describe the relationship between the graph of \( f(x) = x^2 \) and the graph of \( f(x) = -cx^2 \) for any positive constant \( c \)?

74. Error Analysis  A student graphing \( f(x) = |x + 2| + 3 \) obtained the following display. Explain how you can tell by inspection that an error has been made.

-5, 5, 1 by [-2, 8, 1]

12.3 Horizontal and Vertical Translations of the Graphs of Functions

Simplify each expression.

1. \( \frac{x}{x - 2} + \frac{1}{x} \)
2. \( \frac{x}{x - 2} \cdot \frac{1}{x} \)
3. \( \frac{x}{x - 2} + \frac{1}{x} \)
4. \( (x - y)^2 - x^2 - y^2 \)
5. \( 5(2x - 3y) - 4(x - 2y) + 3x - y \)
**Section 12.4**

**Reflecting, Stretching, and Shrinking Graphs of Functions**

**Objectives:**
1. Recognize and use the reflection of a graph.
2. Use stretching and shrinking factors to graph functions.

A key concept of algebra is that there are some basic families of graphs. In Section 12.3, we learned how to shift a graph to create an exact copy of this graph at another location in the plane. We now learn how to modify a given graph to create a stretching or a shrinking of this shape. We start by examining the reflection of a graph.

1. Recognize and Use the Reflection of a Graph

**What Is Meant by the Reflection of a Point?**

The reflection of a point \( (x, y) \) across the \( x \)-axis is a mirror image of the point on the opposite side of the \( x \)-axis. The reflection of the point \( (x, y) \) about the \( x \)-axis is the point \( (x, -y) \).

We now examine the reflection of an entire graph by examining the functions \( y_1 = |x| \) and \( y_2 = -|x| \).

<table>
<thead>
<tr>
<th>Numerically</th>
<th>Graphically</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y_1 =</td>
</tr>
<tr>
<td>-3</td>
<td>3</td>
</tr>
<tr>
<td>-2</td>
<td>2</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
</tbody>
</table>

**Verbally**

Each \( y_2 \)-value is the opposite of the corresponding \( y_1 \)-value. The graph of \( y_2 \) can be obtained by reflecting across the \( x \)-axis each point of the graph of \( y_1 \).

A reflection of a graph is simply a reflection of each point of the graph. The following box describes a reflection of a graph across the \( x \)-axis.

**Reflection of \( y = f(x) \) Across the \( x \)-Axis**

<table>
<thead>
<tr>
<th>Graphically</th>
<th>Numerically</th>
<th>Algebraically</th>
</tr>
</thead>
<tbody>
<tr>
<td>To obtain the graph of ( y = -f(x) ), reflect the graph of ( y = f(x) ) across the ( x )-axis.</td>
<td>For each value of ( x ), ( y_2 ) is the additive inverse of ( y_1 ). That is, ( y_2 = -y_1 ).</td>
<td>Original function: ( y_1 = f(x) ) \hspace{1cm} Reflection: ( y_2 = -f(x) )</td>
</tr>
</tbody>
</table>
Example 1  Examining a Reflection of a Quadratic Function

Compare the functions $y_1 = x^2$ and $y_2 = -x^2$ numerically, graphically, and verbally.

**Solution**

### Numerically

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 = x^2$</th>
<th>$y_2 = -x^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>9</td>
<td>-9</td>
</tr>
<tr>
<td>-2</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>-4</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>-9</td>
</tr>
</tbody>
</table>

### Graphically

For each value of $x$, $y_2 = -y_1$.

The parabola defined by $y_2 = -x^2$ can be obtained by reflecting the graph of $y_1 = x^2$ across the $x$-axis.

**Self-Check 1**

a. Graph $y_1 = x^2 - 1$ and $y_2 = -y_1$ on the same coordinate axes.

b. Write an equation for $y_2$ in terms of $x$.

c. Compare the graphs of $y_1$ and $y_2$.

Example 2 clearly illustrates the reflection idea when we visually compare these two graphs. According to David Bock, graphics research programmer for the National Center for Supercomputing Applications, “This concept is used extensively in three-dimensional computer graphics and animation to create and position symmetrical objects and models.”

Example 2  Forming the Reflection of a Graph

Use the graph of $y_1 = f(x)$ to graph $y_2 = -f(x)$.

**Solution**

Start by reflecting the key points labeled on the graph of $y_1 = f(x)$. Then use the shape of $y_1 = f(x)$ to sketch its reflection across the $x$-axis.
Example 3 is used to examine reflections when two functions are defined by a table of values.

Example 3  Identifying a Reflection Across the x-Axis

Use the table of values for $y_1 = x^3 - x^2$ and $y_2$ to write an equation for $y_2$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 = x^3 - x^2$</th>
<th>$y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−3</td>
<td>−36</td>
<td>36</td>
</tr>
<tr>
<td>−2</td>
<td>−12</td>
<td>12</td>
</tr>
<tr>
<td>−1</td>
<td>−2</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>−4</td>
</tr>
<tr>
<td>3</td>
<td>18</td>
<td>−18</td>
</tr>
</tbody>
</table>

Solution

For each value of $x$, $y_2$ is the additive inverse of $y_1$.

Answer: $y_2 = -y_1$ or $y_2 = -x^3 + x^2$

Self-Check 3

Use the table for $y_1 = f(x)$ to complete the table for $y_2 = -f(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1$</th>
<th>$y_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>−1</td>
<td>−4</td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>9</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>−6</td>
<td></td>
</tr>
</tbody>
</table>

2. Use Stretching and Shrinking Factors to Graph Functions

What Does a Stretching or a Shrinking Factor Do to a Graph?

Stretching or shrinking factors affect the vertical scale of the graph. They do not affect the horizontal scale, and they are not rigid translations. We start our inspection of scaling factors by examining the functions $y_1 = |x|$ and $y_2 = 3|x|$.

| $x$ | $y_1 = |x|$ | $y_2 = 3|x|$ |
|-----|------------|------------|
| −3  | 3          | 9          |
| −2  | 2          | 6          |
| −1  | 1          | 3          |
| 0   | 0          | 0          |
| 1   | 1          | 3          |
| 2   | 2          | 6          |
| 3   | 3          | 9          |

Numerically

Graphically
Verbally

Each value of \( y_2 \) is 3 times the corresponding \( y_1 \)-value.

Comparing the graph of \( y_2 = 3|x| \) to that of \( y_1 = |x| \), we call 3 a **stretching factor** because this factor vertically stretches the graph of \( y_1 = |x| \). The following box describes vertical scaling factors that can either stretch or shrink a graph. Then Example 4 examines a scaling factor of \( \frac{1}{4} \) that vertically shrinks the graph.

Vertical Stretching and Shrinking Factors of \( y = f(x) \)

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Graphically</th>
<th>Numerically</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>If ( c &gt; 1 ):</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Original function: ( y_1 = f(x) )</td>
<td>To obtain the graph of ( y_2 = cy_1 ), vertically stretch the graph of ( y_1 = f(x) ) by a factor of ( c ).</td>
<td></td>
</tr>
<tr>
<td>Scaled function: ( y_2 = cf(x) )</td>
<td>For each value of ( x ), ( y_2 = cy_1 ).</td>
<td></td>
</tr>
<tr>
<td><strong>If ( 0 &lt; c &lt; 1 ):</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Original function: ( y_1 = f(x) )</td>
<td>To obtain the graph of ( y_2 = cy_1 ), vertically shrink the graph of ( y_1 = f(x) ) by a factor of ( c ).</td>
<td></td>
</tr>
<tr>
<td>Scaled function: ( y_2 = cf(x) )</td>
<td>For each value of ( x ), ( y_2 = cy_1 ).</td>
<td></td>
</tr>
</tbody>
</table>

The parabola defined by \( y_2 = \frac{1}{4}x^2 \) can be obtained by vertically shrinking the parabola defined by \( y_1 = x^2 \) by a factor of \( \frac{1}{4} \). At each \( x \)-value, the height of \( y_2 \) is \( \frac{1}{4} \) the height of \( y_1 \). Note that points on the \( x \)-axis are not moved by scaling factors. \( \frac{1}{4}(0) = 0 \), thus \((0, 0)\) is the vertex of both parabolas.

**Example 4**  
**Examining a Shrinking of a Quadratic Function**

Compare the functions \( y_1 = x^2 \) and \( y_2 = \frac{1}{4}x^2 \) numerically, graphically, and verbally.

**Solution**

<table>
<thead>
<tr>
<th>Numerically</th>
<th>Graphically</th>
<th>Verbally</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y_1 = x^2 )</td>
<td>( y_2 = \frac{1}{4}x^2 )</td>
</tr>
<tr>
<td>(-3)</td>
<td>9</td>
<td>( \frac{9}{4} )</td>
</tr>
<tr>
<td>(-2)</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>(-1)</td>
<td>1</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>( \frac{1}{4} )</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>3</td>
<td>9</td>
<td>( \frac{9}{4} )</td>
</tr>
</tbody>
</table>
Solution

For each value of $x$, $y_2 = 2y_1$.

Example 5 Examine a Stretching of a Square Root Function

Compare the functions $y_1 = \sqrt{x}$ and $y_2 = 2\sqrt{x}$ numerically, graphically, and verbally.

Solution

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 = \sqrt{x}$</th>
<th>$y_2 = 2\sqrt{x}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>1</td>
<td>1.00</td>
<td>2.00</td>
</tr>
<tr>
<td>2</td>
<td>1.41</td>
<td>2.83</td>
</tr>
<tr>
<td>3</td>
<td>1.73</td>
<td>3.46</td>
</tr>
<tr>
<td>4</td>
<td>2.00</td>
<td>4.00</td>
</tr>
<tr>
<td>5</td>
<td>2.24</td>
<td>4.47</td>
</tr>
<tr>
<td>6</td>
<td>2.45</td>
<td>4.90</td>
</tr>
</tbody>
</table>

Verbally:

For each value of $x$, $y_2 = 2y_1$.

Self-Check 5

Use the graph of $y_1 = f(x)$ to graph $y_2 = 2f(x)$. Hint: Start by stretching the labeled points.

Example 6 Identifying a Stretching Factor

Use the table of values for $y_1 = f_1(x)$ and $y_2 = f_2(x)$ to write an equation for $y_2$ in terms of $f_1(x)$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y_1 = f_1(x)$</th>
<th>$y_2 = f_2(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>−2</td>
<td>5</td>
<td>15</td>
</tr>
<tr>
<td>−1</td>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>12</td>
</tr>
</tbody>
</table>

Solution

For each value of $x$, $y_2 = 3y_1$. Thus $y_2 = 3f_1(x)$.

Answer: $y_2 = 3f_1(x)$
12.4 Reflecting, Stretching, and Shrinking Graphs of Functions

Self-Check 6

Use the table for \( y_1 = f(x) \) to write an equation for \( y_2 \) in terms of \( f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_1 )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>-1</td>
<td>3</td>
<td>15</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>-2</td>
<td>-10</td>
</tr>
<tr>
<td>2</td>
<td>-5</td>
<td>-25</td>
</tr>
</tbody>
</table>

Every graph in the family of absolute value functions can be obtained from the V-shaped graph of \( y = |x| \). To obtain other members of this family, we can use horizontal and vertical translations, stretching and shrinking factors, and reflections. In Example 7, a function that both shrinks and reflects \( y = |x| \) is examined.

The graph of \( y_2 = \frac{1}{2}|x| \) can be obtained by vertically shrinking the graph of \( y_1 = |x| \) by a factor of \( \frac{1}{2} \) and then reflecting this graph across the \( x \)-axis. Both graphs have the characteristic V shape. The vertex of both V-shaped graphs is \((0, 0)\), but they open in opposite directions.

Example 7

Combining a Shrinking Factor and a Reflection

Use the graph of \( y_1 = |x| \) to graph \( y_2 = \frac{1}{2}|x| \).

Solution

| \( x \) | \( y_1 = |x| \) | \( y_2 = \frac{1}{2}|x| \) |
|---|---|---|
| -4 | 4 | -2 |
| -2 | 2 | -1 |
| 0 | 0 | 0 |
| 2 | 2 | -1 |
| 4 | 4 | -2 |

Graphically

Verbally

Every \( y_2 \) value can be obtained by multiplying the corresponding \( y_1 \) value by \( \frac{1}{2} \).

\[ y_2 = \frac{1}{2}y_1 \]

Self-Check 7

Compare the graphs of \( y_1 = x^2 \) and \( y_2 = -2x^2 \).

Example 8 illustrates how to analyze a graph that involves horizontal and vertical translations and a reflection of a parabola.
Example 8 Combining Translations and a Reflection

Use the graph of \( y_1 = x^2 \) to write an equation for \( y_2 \). Give the domain and the range of each function.

**Solution**

The vertex of \( y_1 = x^2 \) is \((0, 0)\), and the vertex of \( y_2 = f_2(x) \) is \((2, 1)\). The parabola defined by \( y_2 = f_2(x) \) is the same size as the parabola defined by \( y_1 = x^2 \). There is no stretching or shrinking in this example. The graph of \( y_2 = f_2(x) \) can be obtained from the graph of the parabola defined by \( y_1 = x^2 \) by first shifting this graph to the right 2 units, then reflecting this graph across the \( x \)-axis, and finally shifting this graph up 1 unit.

**Answer:**

For \( y_1 \) the domain is \( D = \mathbb{R} \) and the range is \( R = [0, \infty) \).

For \( y_2 \) the domain is \( D = \mathbb{R} \) and the range is \( R = (-\infty, 1] \).

**Self-Check 8**

Use a reflection and a translation of \( y_1 = x^2 \) to write an equation for \( y_2 \).

We can think of the shifts and reflections following the same order as the order of operations in the expression. For \( y_2 = -(x - 2)^2 + 1 \), we first subtract 2 inside the parentheses to produce a shift 2 units to the right. Next we multiply by \(-1\) to produce a reflection across the \( x \)-axis. Finally we add 1, to produce a shift 1 unit up.

The parabola \( f(x) = x^2 \) has a vertex of \((0, 0)\). Example 9 provides practice identifying the vertex of a parabola from the defining equation.

Example 9 Determining the Vertex of a Parabola

Determine the vertex of each parabola by using the fact that the vertex of \( f(x) = x^2 \) is at \((0, 0)\).

**Solution**

(a) \( f(x) = 7x^2 \)

Vertex of \((0, 0)\)

The stretching factor of 7 does not move the vertex of \( f(x) = x^2 \). Points on the \( x \)-axis are not moved by scaling factors.

(b) \( f(x) = (x + 9)^2 \)

Vertex of \((-9, 0)\)

This is a horizontal shift 9 units left of every point on \( f(x) = x^2 \). The vertex is shifted from \((0, 0)\) to \((-9, 0)\).
In Exercises 1–5 evaluate each expression for \( f(x) = 3x^2 - 4 \).

1. \( f(2) \)  
2. \(-f(2)\)  
3. \( f(-2) \)  
4. \( f(0) \)  
5. \(-f(0)\)

1. Vertical and horizontal shifts of a graph create another graph whose shape is the ______ as that of the original graph and whose size is the ______ as that of the original graph.
2. The reflection of the point \((x, y)\) across the \(x\)-axis is the point ______.
3. If \( y = f(x) \) and \( c > 1 \), then we call \( c \) a ______ factor in the function \( y = cf(x) \).
4. If \( y = f(x) \) and \( 0 < c < 1 \), then we call \( c \) a ______ factor in the function \( y = cf(x) \).
5. If \( y = f(x) \) and \( c = -1 \), then we call the graph of \( y = cf(x) \) a ______ of the graph of \( y = f(x) \) across the ______-axis.
6. If we examine the shapes of \( y = f(x) \) and \( y = 2f(x) \), then we will observe that the two graphs have the ______ basic shape but that \( y = 2f(x) \) is obtained by vertically ______ \( y = f(x) \) by a factor of ______.
**12.4 Exercises**

**Objective 1 Recognize and Use the Reflection of a Graph**

Exercises 1–8 give some basic functions and reflections of these functions. Match each graph to the correct function.

1. \( f(x) = |x| \)
2. \( f(x) = -|x| \)
3. \( f(x) = x \)
4. \( f(x) = -x \)
5. \( f(x) = \sqrt{x} \)
6. \( f(x) = -\sqrt{x} \)
7. \( f(x) = x^3 \)
8. \( f(x) = -x^3 \)

![Graphs A, B, C, D](image1)

In Exercises 9 and 10, complete the given table of values.

9. \[
\begin{array}{c|c|c}
   x & f(x) & -f(x) \\
   \hline
   0 & 4 & \hfill \\
   1 & 2 & -3 \hfill \\
   2 & 2 & -1 \\
   3 & 4 & 0 \\
   4 & 6 & 1 \\
\end{array}
\]

10. \[
\begin{array}{c|c|c}
   x & f(x) & -f(x) \\
   \hline
   0 & -3 & 5 \\
   1 & -2 & 0 \\
   2 & -1 & 3 \\
   3 & 0 & -4 \\
   4 & 1 & -3 \\
\end{array}
\]

In Exercises 11 and 12, use the graph of \( y = f(x) \) to graph \( y = -f(x) \).

11. ![Graph E](image2)

12. ![Graph F](image3)

**Objective 2 Use Stretching and Shrinking Factors to Graph Functions**

Exercises 13–16 give functions that are obtained by either stretching or shrinking the graph of \( f(x) = x^2 \). Match each graph to the correct function.

13. \( f(x) = 2x^2 \)
14. \( f(x) = 5x^2 \)
15. \( f(x) = \frac{1}{2}x^2 \)
16. \( f(x) = \frac{1}{5}x^2 \)

![Graphs A, B, C, D](image4)
Exercises 17–20 give functions that are obtained by reflecting across the x-axis or stretching or shrinking the graph of \( f(x) = x \). Match each graph to the correct function.

17. \( f(x) = 5x \)

A.

18. \( f(x) = \frac{1}{5}x \)

B.

19. \( f(x) = -2x \)

C.

20. \( f(x) = -\frac{1}{2}x \)

D.

Exercises 21–24 give functions that are obtained by either stretching or shrinking the graph of \( f(x) = |x| \). Match each graph to the correct function.

21. \( f(x) = \frac{3}{4}|x| \)

A.

22. \( f(x) = 3|x| \)

B.

23. \( f(x) = \frac{3}{2}|x| \)

C.

24. \( f(x) = \frac{1}{4}|x| \)

D.

Exercises 25–28 give functions that are obtained by either stretching or shrinking the graph of \( y = \sqrt{x} \). Match each graph to the correct function.

25. \( f(x) = 4\sqrt{x} \)

A.

26. \( f(x) = \frac{1}{4}\sqrt{x} \)

B.

27. \( f(x) = \frac{3}{4}\sqrt{x} \)

C.

28. \( f(x) = \frac{9}{4}\sqrt{x} \)

D.

Skill and Concept Development

In Exercises 29–34, use the given graph of \( y = f(x) \) to graph each function. (Hint: First graph the three key points on the new graph.)

29. \( y = -f(x) \)

30. \( y = \frac{1}{2}f(x) \)

31. \( y = 2f(x) \)

32. \( y = -3f(x) \)

33. \( y = f(x) + 2 \)

34. \( y = f(x + 2) \)

In Exercises 35–38, use the given table of values for \( y = f(x) \) to complete each table.

\[
\begin{array}{c|c}
  x & f(x) \\
-2 & 18 \\
-1 & 9 \\
0 & 3 \\
1 & 1 \\
2 & 3 \\
\end{array}
\]

\[
\begin{array}{c|c}
  x & \frac{1}{3}f(x) \\
-2 & \\
-1 & \\
0 & 1 \\
1 & 1 \\
2 & 2 \\
\end{array}
\]
In Exercises 39–46, use the given table to write an equation for \( y_2 \) in terms of \( f(x) \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>12</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>2</td>
<td>-1.5</td>
</tr>
<tr>
<td>3</td>
<td>-3</td>
</tr>
<tr>
<td>4</td>
<td>-5</td>
</tr>
<tr>
<td>5</td>
<td>-6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
</tr>
<tr>
<td>2</td>
<td>-2</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>12</td>
</tr>
<tr>
<td>5</td>
<td>16</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-3</td>
<td>8</td>
</tr>
<tr>
<td>-2</td>
<td>3</td>
</tr>
<tr>
<td>-1</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>12</td>
</tr>
<tr>
<td>1</td>
<td>20</td>
</tr>
<tr>
<td>2</td>
<td>24</td>
</tr>
</tbody>
</table>

In Exercises 47–54, determine the vertex of each parabola by using the fact that the vertex of \( f(x) = x^2 \) is at \( (0, 0) \).

47. \( f(x) = 8x^2 \)
48. \( f(x) = \frac{1}{2}x^2 \)
49. \( f(x) = 8x^2 - 7 \)
50. \( f(x) = \frac{1}{7}(x - 7)^2 \)
51. \( f(x) = (x - 8)^2 - 7 \)
52. \( f(x) = (x + 6)^2 + 9 \)
53. \( f(x) = -(x - 8)^2 + 7 \)
54. \( f(x) = -(x + 6)^2 - 9 \)

55. Translate or reflect the graph of \( f(x) = x^2 \) as described by each equation.
   a. Graph \( f(x) = (x - 3)^2 \), and describe how to obtain this graph from \( f(x) = x^2 \).
   b. Graph \( f(x) = -(x - 3)^2 \), and describe how to obtain this graph from \( f(x) = (x - 3)^2 \).
   c. Graph \( f(x) = -(x - 3)^2 + 2 \), and describe how to obtain this graph from \( f(x) = -(x - 3)^2 \).

56. Translate or reflect the graph of \( f(x) = |x| \) as described by each equation.
   a. Graph \( f(x) = |x + 2| \), and describe how to obtain this graph from \( f(x) = |x| \).
   b. Graph \( f(x) = -|x + 2| \), and describe how to obtain this graph from \( f(x) = |x + 2| \).
   c. Graph \( f(x) = -|x + 2| - 3 \), and describe how to obtain this graph from \( f(x) = -|x + 2| \).

Exercises 57–62 describe a translation, a reflection, a stretching, or a shrinking of \( y = f(x) \). Match each description to the correct function.

57. A vertical stretching of \( y = f(x) \) by a factor of 7
   A. \( y = f(x + 7) \)
   B. \( y = 7f(x) \)

58. A vertical shrinking of \( y = f(x) \) by a factor of \( \frac{1}{7} \)
   C. \( y = \frac{1}{7}f(x) \)

59. A reflection of \( y = f(x) \) across the x-axis
   D. \( y = -f(x) \)
   E. \( y = f(x) + 7 \)
   F. \( y = f(x - 7) \)

60. A horizontal shift of \( y = f(x) \) left 7 units

61. A horizontal shift of \( y = f(x) \) right 7 units

62. A vertical shift of \( y = f(x) \) up 7 units

In Exercises 63–68, determine the range of each function, given that the range of \( y = f(x) \) is \([2, 6]\).

63. \( y = f(x) + 2 \)
64. \( y = f(x) - 2 \)
65. \( y = 2f(x) \)
66. \( y = \frac{1}{2}f(x) \)
67. \( y = -f(x) \)
68. \( y = -3f(x) \)

69. Write the first five terms of each sequence. (Hint: See Example 6 in Section 1.7.)
   a. \( a_n = n^3 \)
   b. \( a_n = -n^3 \)
   c. \( a_n = 2n^3 \)
   d. \( a_n = n^3 - 2 \)

70. Write the first five terms of each sequence.
   a. \( a_n = |n| \)
   b. \( a_n = |n - 5| \)
   c. \( a_n = -|n - 5| \)
   d. \( a_n = 2|n - 5| \)

Connecting Concepts to Applications

71. Sequence of Retirement Bonuses As part of a bonus plan, a company gives each secretary a number of shares in the company at retirement. The number of shares given equals the number of years the secretary has worked.
a. Write a formula for $A_n$, the sequence of the number of shares that would be given for a retirement after $n$ years.

b. Write a formula for $V_n$, the value of the shares that would be given for a retirement after $n$ years if the value of each share is $50$.

72. Comparing the Distance Traveled by Two Airplanes

The formula $D = RT$ can be used to determine the distance $D$ flown by an airplane flying at a rate $R$ for time $T$. Complete the table if the rate of the second plane is double that of the first plane.

<table>
<thead>
<tr>
<th>Plane 1</th>
<th>Plane 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hours $T$</td>
<td>Distance $D$</td>
</tr>
<tr>
<td>1</td>
<td>200</td>
</tr>
<tr>
<td>2</td>
<td>400</td>
</tr>
<tr>
<td>3</td>
<td>600</td>
</tr>
<tr>
<td>4</td>
<td>800</td>
</tr>
</tbody>
</table>

73. Comparing the Growth of Two Investments

The formula $I = PRT$ can be used to determine the interest earned on an investment of $P$ dollars for $T$ years at simple interest rate $R$. A second investment of the same amount is invested at a rate that is three-fourths that of the first investment. Complete the table for the second investment.

<table>
<thead>
<tr>
<th>Investment 1</th>
<th>Investment 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Years $T$</td>
<td>Interest $I$</td>
</tr>
<tr>
<td>5</td>
<td>4,000</td>
</tr>
<tr>
<td>10</td>
<td>8,000</td>
</tr>
<tr>
<td>15</td>
<td>12,000</td>
</tr>
<tr>
<td>20</td>
<td>16,000</td>
</tr>
</tbody>
</table>

74. Comparing the Production of Two Assembly Lines

The following graph shows the number of lightbulbs produced by the workers at an assembly line during a very busy day shift. The night shift, which has fewer workers than the day shift, produces one-third as many bulbs as the day shift. Sketch the graph of the number of lightbulbs produced by the workers on the night shift.

75. Discovery Question

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-2$</td>
<td>11</td>
</tr>
<tr>
<td>$-1$</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>$-1$</td>
</tr>
<tr>
<td>1</td>
<td>$-4$</td>
</tr>
<tr>
<td>2</td>
<td>$-5$</td>
</tr>
<tr>
<td>3</td>
<td>$-4$</td>
</tr>
<tr>
<td>4</td>
<td>$-1$</td>
</tr>
</tbody>
</table>

76. Discovery Question

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>0.5</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-2.0$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$-3.5$</td>
</tr>
<tr>
<td>0</td>
<td>$-4.0$</td>
</tr>
<tr>
<td>1</td>
<td>$-3.5$</td>
</tr>
<tr>
<td>2</td>
<td>$-2.0$</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
</tr>
</tbody>
</table>

77. Error Analysis

A student graphing $y = 0.5|x|$ obtained the following display. Explain how you can tell by inspection that an error has been made.

78. Error Analysis

A student graphing $y = -3\sqrt{x}$ obtained the following display. Explain how you can tell by inspection that an error has been made.
1. Write the equation of the horizontal line through (2, -3).
2. Write the equation of the vertical line through (2, -3).
3. Write in slope-intercept form the equation of the line through (2, -3) with slope $m = \frac{3}{5}$.
4. Write in slope-intercept form the equation of the line through (2, -3) and (7, 2).
5. A line with slope $m = \frac{3}{4}$ passes through (2, -3) and through another point with an $x$-coordinate of 6. What is the $y$-coordinate of this point?

### Example 1: Determining the Sum of Two Functions

Find the sum of $f(x) = x^2$ and $g(x) = 2$ algebraically, numerically, and graphically.

**Solution**

**Algebraically**

\[(f + g)(x) = f(x) + g(x)\]
\[(f + g)(x) = x^2 + 2\]

**Numerically**

<table>
<thead>
<tr>
<th>$x$</th>
<th>$f(x) = x^2$</th>
<th>$g(x) = 2$</th>
<th>$(f + g)(x) = x^2 + 2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-2</td>
<td>4</td>
<td>-2</td>
<td>6</td>
</tr>
<tr>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
<td>6</td>
</tr>
</tbody>
</table>

The new function called $f + g$ is determined by adding $f(x)$ and $g(x)$. Substitute $x^2$ for $f(x)$ and 2 for $g(x)$. This table contains only a few values from the domain of input values from $\mathbb{R}$. These values illustrate that $(f + g)(x)$ can be determined by adding $f(x)$ and $g(x)$ for each input: $4 + 2 = 6$, $1 + 2 = 3$, $0 + 2 = 2$, $1 + 2 = 3$, and $4 + 2 = 6$. 

### Algebra of Functions

**Objectives:**

1. Add, subtract, multiply, and divide two functions.
2. Form the composition of two functions.

Problems, such as those in business, are often broken down into simpler components for analysis. For example, to determine the profit made by producing and selling an item, both the revenue and the cost must be known. Separate divisions of a business might be asked to find the revenue function and the cost function; the profit function would then be found by properly combining these two functions. We shall examine five ways to combine functions: sum, difference, product, quotient, and composition of functions.

1. **Add, Subtract, Multiply, and Divide Two Functions**

The sum of two functions $f$ and $g$, denoted by $f + g$, is defined as

\[(f + g)(x) = f(x) + g(x)\]

for all values of $x$ that are in the domain of both $f$ and $g$. Note that if either $f(x)$ or $g(x)$ is undefined, then $(f + g)(x)$ is also undefined.

**What Do I Get When I Add Two Functions?**

You get another function. This is consistent with earlier additions. When we add two real numbers, we get another real number. When we add two polynomials, we get another polynomial.
12.5 Algebra of Functions

The graph of \( f(x) = x^2 \) is a parabola opening upward with its vertex at \((0, 0)\).
The graph of \( g(x) = 2 \) is the horizontal line representing a constant output of 2.
The new function \( f + g \) adds all output values. Since \( g(x) = 2 \), all output values of \( f \) are translated up 2 units when \( g(x) \) is added to \( f(x) \).

### Self-Check 1

Given \( f(x) = x^2 \) and \( g(x) = -4x + 4 \), determine \( f + g \).

#### When I Add Two Functions, How Can I Check My Answer?

One way is to compare the table of values for the original problem with the table of values for the sum that is determined. If two functions are equal, their input and output values are identical. If the domain of input values is the set of all real numbers, then we cannot list all the input-output pairs. However, a table of values can serve to check that two functions yield the same values.

The **difference of two functions** \( f \) and \( g \), denoted by \( f - g \), is defined as

\[
(f - g)(x) = f(x) - g(x)
\]

for all values of \( x \) that are in the domain of both \( f \) and \( g \).

#### Example 2 Using the Difference of Two Functions to Model Profit

Suppose that the weekly revenue function for \( u \) units of a product sold is \( R(u) = 5u^2 - 7u \) dollars, and the cost function for \( u \) units is \( C(u) = 8u + 23 \). The fewest number of units that can be produced is 0 and 100 is the greatest number that can be marketed. Assuming profit can be determined by subtracting the cost from the revenue, find the profit function \( P \) and determine \( P(4) \), the profit made by selling 4 units.

#### Solution

**Word Equation**

Profit = Revenue − Cost

First write a word equation to model this problem.

**Algebraic Equation**

\[
\begin{align*}
P(u) &= R(u) - C(u) \\
P(u) &= (5u^2 - 7u) - (8u + 23) \\
P(u) &= 5u^2 - 7u - 8u - 23 \\
P(u) &= 5u^2 - 15u - 23
\end{align*}
\]

Substitute the given expressions for \( R(u) \) and \( C(u) \).

The domain is \( 0 \leq u \leq 100 \). (For some types of units, \( u \) may have to be an integer.)

This is the profit function.
The operations of multiplication and division are defined similarly to addition except that the quotient is not defined if $g(x) = 0$.

The domain of all these functions except $\frac{f}{g}$ is the set of values in both the domain of $f$ and the domain of $g$. For $\frac{f}{g}$, we also must have $g(x) \neq 0$.

### Operations on Functions

<table>
<thead>
<tr>
<th>Notation</th>
<th>Definition</th>
<th>Examples: For $f(x) = x^2 - 4$ and $g(x) = x - 2$:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sum</td>
<td>$f + g$</td>
<td>$(f + g)(x) = f(x) + g(x)$</td>
</tr>
<tr>
<td>Difference</td>
<td>$f - g$</td>
<td>$(f - g)(x) = f(x) - g(x)$</td>
</tr>
<tr>
<td>Product</td>
<td>$f \cdot g$</td>
<td>$(f \cdot g)(x) = f(x) \cdot g(x)$</td>
</tr>
<tr>
<td>Quotient</td>
<td>$\frac{f}{g}$</td>
<td>$(\frac{f}{g})(x) = \frac{f(x)}{g(x)}$</td>
</tr>
</tbody>
</table>

### Example 3  Determining the Product and Quotient of Two Functions

Given $f(x) = x^2 + 5x$ and $g(x) = \frac{x + 5}{x}$, find the following.

(a) $(f \cdot g)(x)$  (b) The domain of $(f \cdot g)(x)$
(c) $(\frac{f}{g})(x)$  (d) The domain of $(\frac{f}{g})(x)$

#### Solution

(a) $(f \cdot g)(x) = f(x) \cdot g(x)$

$(f \cdot g)(x) = (x^2 + 5x)\left(\frac{x + 5}{x}\right)$

$(f \cdot g)(x) = (x + 5)^2$

(b) $f$ is defined for $\mathbb{R}$.

$g$ is defined for $x \neq 0$.

$f \cdot g$ is defined for $x \neq 0$.
Functions are defined not only by their formulas, but also by the set of input values contained in the domains of the functions. We often allow the domain of a function to be implied by the formula. In this case, the domain is understood to be all real numbers for which the formula is defined and produces real number outputs. When we combine functions to produce new functions, we must take care that the formulas are used only for input values that are allowed for the new function. This is illustrated by Example 4. Two functions \( f \) and \( g \) are equal if the domain of \( f \) equals the domain of \( g \) and for each \( x \) in their common domain.

(c) \[ \left( \frac{f}{g} \right)(x) = \frac{f(x)}{g(x)} \]
\[ \left( \frac{f}{g} \right)(x) = \frac{x^2 + 5x}{x + 5} \]
\[ \left( \frac{f}{g} \right)(x) = \frac{x(x + 5)}{1} \cdot \frac{x}{x + 5} \]
To simplify this complex rational expression, invert the divisor and multiply.
\[ \left( \frac{f}{g} \right)(x) = x^2 \]

(d) \( f \) is defined for \( \mathbb{R} \).
\( g \) is defined for \( x \neq 0 \).
\( g(x) = 0 \) for \( x = -5 \).
\( \frac{f}{g} \) is defined for \( x \neq 0 \) and \( x \neq -5 \).

Self-Check 3
Given \( f(x) = x^2 - 1 \) and \( g(x) = x^2 + 1 \), determine these two functions.

a. \( (f \cdot g)(x) \)  

b. \( \left( \frac{f}{g} \right)(x) \)

Functions are defined not only by their formulas, but also by the set of input values contained in the domains of the functions. We often allow the domain of a function to be implied by the formula. In this case, the domain is understood to be all real numbers for which the formula is defined and produces real number outputs. When we combine functions to produce new functions, we must take care that the formulas are used only for input values that are allowed for the new function. This is illustrated by Example 4. Two functions \( f \) and \( g \) are equal if the domain of \( f \) equals the domain of \( g \) and \( f(x) = g(x) \) for each \( x \) in their common domain.

Example 4  Comparing Functions to Determine Whether They Are Equal

Given \( f(x) = x \), \( g(x) = \frac{x^3 - 4x}{x^2 - 4} \), and \( h(x) = \frac{x^3 + 4x}{x^2 + 4} \), determine whether:

(a) \( f = g \)  

(b) \( f = h \)

Solution

(a) \( f(x) = x \) for all real \( x \)
\( g(x) = \frac{x^3 - 4x}{x^2 - 4} \) for \( x \neq \pm 2 \) Both \(-2\) and \(2\) result in division by 0.
\( g(x) = \frac{x(x^2 - 4)}{x^2 - 4} \) for \( x \neq \pm 2 \)
\( g(x) = x \) for \( x \neq \pm 2 \)
\( f \neq g \) because \( f \) is defined for \( x = -2 \) and \( x = 2 \) but \( g \) is not.
\( f(x) \) and \( g(x) \) have the same values for all real numbers except \( x = -2 \) and \( x = 2 \).
(b) \( f(x) = x \) for all real \( x \)

\[
h(x) = \frac{x^3 + 4x}{x^2 + 4} \quad \text{for all real } x
\]

There are no real numbers for which the denominator of \( h \) is 0.

\[ h(x) = \frac{x(x^2 + 4)}{x^2 + 4} \quad \text{for all real } x \]

\[ h(x) = x \quad \text{for all real } x \]

\( f = h \) because these functions produce the same output for each real number \( x \).

**Self-Check 4**

Explain why \( f(x) = \frac{x^2 - 4x + 4}{5x - 10} \) is not equal to \( g(x) = \frac{x - 2}{5} \).

As shown in the following figure, the graphs of the three functions \( f, g, \) and \( h \) in Example 4 are nearly identical. The only difference is that the graph of \( g \) has “holes” at \( x = -2 \) and \( x = 2 \) because these values are not in its domain.

### 2. Form the Composition of Two Functions

Functions, especially in formula form, are a powerful means of describing the relationship between two quantities. We can further amplify this power by “chaining” two functions together. This way of combining two functions is called **composition**.

**What Symbol Do I Use to Denote the Composition of Two Functions?**

The symbol \( \circ \) denotes the composition of two functions. The notation \( f \circ g \) should not be confused with the symbol for the multiplication of two functions \( f \cdot g \).

### Composite Function \( f \circ g \)

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Verbally</th>
<th>Algebraic Example</th>
</tr>
</thead>
</table>
| \( (f \circ g)(x) = f(g(x)) \) | \( f \circ g \) denotes the composition of function \( f \) with function \( g \). \( f \circ g \) is read “\( f \) composed with \( g \)” | For \( f(x) = x + 1 \) and \( g(x) = 2x \):
   \[
   (f \circ g)(x) = f(g(x)) = f(2x) = 2x + 1
   \] |
| The domain of \( f \circ g \) is the set of \( x \)-values from the domain of \( g \) for which \( g(x) \) is in the domain of \( f \). | | |

Functions can be examined by using mapping notation, ordered-pair notation, tables, graphs, and function notation. In each case, to evaluate \( (f \circ g)(x) \), we first evaluate \( g(x) \) and then apply \( f \) to this result.
Example 5  Determining the Composition of Two Functions

Find \( f \circ g \) for the given functions \( f \) and \( g \).

**Solution**

<table>
<thead>
<tr>
<th>( g )</th>
<th>( f )</th>
<th>Verbally</th>
<th>Mapping Notation</th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td>6 ( \rightarrow ) 3</td>
<td>3 ( \rightarrow ) 9</td>
<td>( g ) maps 6 to 3; then ( f ) maps 3 to 9.</td>
<td>( f \circ g )</td>
<td>( f \circ g )</td>
</tr>
<tr>
<td>5 ( \rightarrow ) 4</td>
<td>4 ( \rightarrow ) 7</td>
<td>( g ) maps 5 to 4; then ( f ) maps 4 to 7.</td>
<td></td>
<td></td>
</tr>
<tr>
<td>-1 ( \rightarrow ) 0</td>
<td>0 ( \rightarrow ) 2</td>
<td>( g ) maps -1 to 0; then ( f ) maps 0 to 2.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

**Self-Check 5**

The function \( f \) maps 10 to 8 and the function \( g \) maps 4 to 10. What do we know about \( f \circ g \)?

In Example 6, \( f \circ g \neq g \circ f \). Although \( f \circ g \) can equal \( g \circ f \) in special cases, in general the order in which we perform composition of functions is important.

Example 6  Determining the Composition of Two Functions

Given \( f(x) = x^2 \) and \( g(x) = 3x \), evaluate these expressions.

(a) \( (f \circ g)(4) \)  
(b) \( (g \circ f)(4) \)  
(c) \( (f \circ g)(x) \)  
(d) \( (g \circ f)(x) \)

**Solution**

(a) \( (f \circ g)(4) = f[g(4)] = f[3(4)] \)  
\( = f(12) \)  
\( = (12)^2 \)  
\( = 144 \)  
First apply the formula for \( g(x) \). Evaluate \( g(4) = 3(4) \).  
Then apply the formula for \( f(x) \) to 12.

(b) \( (g \circ f)(4) = g[f(4)] = g(4^2) \)  
\( = g(16) \)  
\( = 3(16) \)  
\( = 48 \)  
First apply the formula for \( f(x) \). Evaluate \( f(4) = 4^2 \).  
Then apply the formula for \( g(x) \) to 16.

(c) \( (f \circ g)(x) = f[g(x)] \)  
\( = f(3x) \)  
\( = (3x)^2 \)  
\( = 9x^2 \)  
First apply the formula for \( g(x) \) to \( x \).  
Then apply the formula for \( f(x) \) to \( 3x \).

(d) \( (g \circ f)(x) = g[f(x)] = g(x^2) \)  
\( = 3(x^2) \)  
\( = 3x^2 \)  
First apply the formula for \( f(x) \) to \( x \).  
Then apply the formula for \( g(x) \) to \( x^2 \).

**Self-Check 6**

Given \( f(x) = 3x - 1 \) and \( g(x) = \frac{x + 1}{3} \), evaluate these expressions.

(a) \( (f \circ g)(2) \)  
(b) \( (g \circ f)(2) \)
The functions \( f(x) = 3x - 1 \) and \( g(x) = \frac{x + 1}{3} \) from Self-Check 6 are inverses of each other. Note that

\[
(f \circ g)(2) = 2 \quad \frac{g}{11568} 1 \quad \frac{f}{2} \quad 2 \quad \text{and} \quad (g \circ f)(2) = 2 \quad \frac{f}{5} \quad \frac{g}{2}
\]

For any input value \( x \),

\[
(f \circ g)(x) = f[g(x)] = f\left(\frac{x + 1}{3}\right) = 3\left(\frac{x + 1}{3}\right) - 1 = (x + 1) - 1 = x
\]

Likewise, \((g \circ f)(x) = x\). Thus the composition of functions gives us another way to characterize the inverse of a function.

### Inverse of a Function

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>The functions ( f ) and ( f^{-1} ) are inverses of each other if and only if ((f \circ f^{-1})(x) = x) for each input value of ( f^{-1} )</td>
<td>( f(x) = 3x - 1 ) and ( f^{-1}(x) = \frac{x + 1}{3} ) are inverses because ((f \circ f^{-1})(x) = f\left(\frac{x + 1}{3}\right))</td>
</tr>
<tr>
<td>((f \circ f^{-1})(x) = f\left(\frac{x + 1}{3}\right))</td>
<td>((f \circ f^{-1})(x) = 3\left(\frac{x + 1}{3}\right) - 1)</td>
</tr>
<tr>
<td>((f \circ f^{-1})(x) = x) and ((f^{-1} \circ f)(x) = f^{-1}(3x - 1))</td>
<td>((f^{-1} \circ f)(x) = \frac{(3x - 1) + 1}{3})</td>
</tr>
<tr>
<td>((f^{-1} \circ f)(x) = x)</td>
<td>((f^{-1} \circ f)(x) = x)</td>
</tr>
</tbody>
</table>

This means that we can use tables to confirm that one function is the inverse of another. If we let \( y_1 = f(x) \) and \( y_2 = g(x) \), then by letting \( y_3 = y_1(y_2) \) we are representing \( f[g(x)] \). Showing that \( y_3 \) has the same values as \( x \) confirms that \((f \circ g)(x) = x\).

Example 7 illustrates how a problem can be broken down into pieces with individual functions as mathematical models. Then the overall relationship can be modeled by a composite function.

### Example 7  Composing Cost and Production Functions

The quantity of items a factory can produce weekly is a function of the number of hours it operates. For one company this is given by \( q(t) = 40t \) for \( 0 \leq t \leq 168 \). The dollar cost of manufacturing these items is a function of the quantity produced; in this case, \( C(q) = q^2 - 40q + 750 \) for \( q \geq 0 \). Evaluate and interpret the following expressions.

\( a) \quad q(8) \quad b) \quad C(320) \quad c) \quad (C \circ q)(8) \quad d) \quad (C \circ q)(t) \)

**Solution**

\( a) \quad q(t) = 40t \quad \text{Substitute 8 into the formula for } q(t). \)

\( q(8) = 40(8) \)

\( q(8) = 320 \)

320 units can be produced in 8 hours.
(b) \( C(q) = q^2 - 40q + 750 \)
\[
C(320) = (320)^2 - 40(320) + 750 \\
C(320) = 90,350 \\
\$90,350 is the cost of manufacturing 320 units.
\]
(c) \( (C \circ q)(8) = C(q(8)) \)
\[
(C \circ q)(8) = C(320) \\
(C \circ q)(8) = $90,350 \\
\$90,350 is the cost of 8 hours of production.
\]
(d) \( (C \circ q)(t) = C[q(t)] \)
\[
(C \circ q)(t) = C[40t] \\
(C \circ q)(t) = (40t)^2 - 40(40t) + 750 \\
(C \circ q)(t) = 1,600t^2 - 1,600t + 750 \\
This is the cost of \( t \) hours of production.
\]

**Self-Check 7**

In Example 7, assume that \( q(t) = 50t \) and \( C(q) = q^2 - 40q + 800 \).

a. Determine \( (C \circ q)(t) \).  

b. Determine \( (C \circ q)(10) \).

Example 8 illustrates the analysis of a geometric problem by using a composite function.

**Example 8**  
**Composing Area and Length Functions**

A piece of wire 20 m long is cut into two pieces. The length of the shorter piece is \( s \) m, and the length of the longer piece is \( L \) m. The longer piece is then bent into the shape of a square of area \( A \) m².

(a) Express the length \( L \) as a function of \( s \).

(b) Express the area \( A \) as a function of \( L \).

(c) Express the area \( A \) as a function of \( s \).

**Solution**

(a) \[
L(s) = \frac{20 - s}{4} \\
\text{Length of the longer piece} = \text{Total length} - \text{Length of the shorter piece} \\
\]

(b) \[
A(L) = \left(\frac{L}{4}\right)^2 \\
\text{Area = Square of length of one side} \\
\]

(c) \[
(A \circ L)(s) = A[L(s)] \\
(A \circ L)(s) = A(20 - s) \\
(A \circ L)(s) = \left(\frac{20 - s}{4}\right)^2 \\
\text{The length of one side of the square is one-fourth the perimeter.} \\
\text{First write a word equation that describes the relationship between the length of the longer piece and the length of the shorter piece.} \\
\text{This equation expresses the length as a function of} \ s \ \text{where} \ 0 < s < 10. \\
\text{The composition} \ A \circ L \ \text{expresses} \ A \ \text{as a function of} \ s. \\
\text{Substitute} \ 20 - s \ \text{for} \ L(s) \ \text{from part (a).} \\
\text{Evaluate} \ A \ \text{by using the formula from part (b).} \\
\]
The product of two functions
3. a.

The sum of two functions
1. b.

2. c.

The difference of two functions

Using the Language and Symbolism of Mathematics

12.5

Express the area

b. Express the area

bent into the shape of a rectangle whose length is twice its width.

a. Express the length
of the wire that remains as a function of
x.

b. Express the area
of the rectangle that is formed as a function of
L.

c. Express the area
of the rectangle that is formed as a function of
x.

An important skill in calculus is the ability to take a given function and decompose it into simpler components.

Example 9
Decomposing Functions into Simpler Components

Express each of these functions in terms of
f(x) = 2x - 3 and
\( g(x) = \sqrt{x} \).

(a) \( h(x) = \sqrt{2x - 3} = \sqrt{f(x)} \)

(b) \( h(x) = 2\sqrt{x} = 2g(x) = 3 \)

Solution

(a) \( h(x) = \sqrt{2x - 3} = \sqrt{f(x)} \)

First substitute \( f(x) = 2x - 3 \).

Then replace the square root function with the \( g \) function.

(b) \( h(x) = 2\sqrt{x} = 2g(x) = 3 \)

First substitute \( g(x) = \sqrt{x} \).

Then rewrite the expression by using the definition of the \( f \) function.

Self-Check 8

A piece of wire that is 48 cm long has a piece \( x \) cm cut off. The remaining piece is bent into the shape of a rectangle whose length is twice its width.

a. Express the length \( L \) of the wire that remains as a function of \( x \).

b. Express the area \( A \) of the rectangle that is formed as a function of \( x \).

c. Express the area \( A \) of the rectangle that is formed as a function of \( x \).

Self-Check Answers

1. \( (f + g)(x) = x^2 - 4x + 4 \)

2. \( (f - g)(x) = x^2 - 5x - 2 \)

3. \( f \cdot g \) is undefined for all values of \( x \) that are in the domain of both \( f \) and \( g \).

4. The domain of \( g \) is \( \mathbb{R} \) but the domain of \( f \) is \( \mathbb{R} \sim \{2\} \).

5. \( f \cdot g \) maps 4 to 8.

6. a. \( 2 \)

b. \( 2 \)

7. a. \( (C \cdot g)(t) = 2.500t^2 - 2.000t + 800 \)

This is the cost of \( t \) hours or production.

b. \( (C \cdot g)(10) = 230,800 \)

\$230,800 is the cost of 10 hours of production.

8. a. \( L(x) = 48 - x \)

b. \( A(L) = \frac{t^2}{18} \)

C. \( A(x) = \frac{(48 - x)^2}{18} \)

d. \( h(x) = (g \cdot f)(x) \)

Self-Check 9

Express \( h(x) = \frac{1}{x - 6} \) in terms of \( f(x) = x - 6 \) and \( g(x) = \frac{1}{x} \).

1. The sum of two functions \( f \) and \( g \), denoted by \( f + g \), is defined as \( (f + g)(x) = \) for all values of \( x \) that are in the domain of both \( f \) and \( g \).

2. The difference of two functions \( f \) and \( g \), denoted by \( f - g \), is defined as \( (f - g)(x) = \) for all values of \( x \) that are in the domain of both \( f \) and \( g \).

3. The product of two functions \( f \) and \( g \), denoted by \( f \cdot g \), is defined as \( (f \cdot g)(x) = \) for all values of \( x \) that are in the domain of both \( f \) and \( g \).

4. The quotient of two functions \( f \) and \( g \), denoted by \( \frac{f}{g} \), is defined as \( \left(\frac{f}{g}\right)(x) = \) for all values of \( x \) that are in the domain of both \( f \) and \( g \), provided \( g(x) \neq \) for all values of \( x \) that are in the domain of both \( f \) and \( g \).

5. The composition of the function \( f \) with the function \( g \), denoted by \( f \circ g \), is defined by \( (f \circ g)(x) = \) for all values of \( x \) that are in the domain of both \( f \) and \( g \).
6. The domain of \( f \circ g \) is the set of \( x \)-values from the domain of \( g \) for which \( \underline{\quad} \) is in the domain of \( \underline{\quad} \).

7. The functions \( f \) and \( f^{-1} \) are inverses of each other if and only if \( (f \circ f^{-1})(x) = \underline{\quad} \) for each input value from the domain of \( \underline{\quad} \) and \( (f^{-1} \circ f)(x) = \underline{\quad} \) for each input value from the domain of \( \underline{\quad} \).

### Problem Set

#### Quick Review

1. Add \( (2x^2 - 7x - 15) + (2x + 3) \).
2. Subtract \( (2x^2 - 7x - 15) - (2x + 3) \).
3. Multiply \( (2x^2 - 7x - 15)(2x + 3) \).

The functions \( f \) and \( f^{-1} \) are inverses of each other if and only if \( (f \circ f^{-1})(x) = \underline{\quad} \) for each input value from the domain of \( \underline{\quad} \) and \( (f^{-1} \circ f)(x) = \underline{\quad} \) for each input value from the domain of \( \underline{\quad} \).

### Exercises

#### Objective 1 Add, Subtract, Multiply, and Divide Two Functions

In Exercises 1 and 2, evaluate each expression, given 
\[ f(x) = x^2 - 1 \text{ and } g(x) = 2x + 5. \]

1. a. \( (f + g)(2) \)
   b. \( (f - g)(2) \)
   c. \( (f \cdot g)(2) \)
   d. \( \left(\frac{f}{g}\right)(2) \)

2. a. \( (f + g)(-3) \)
   b. \( (f - g)(-3) \)
   c. \( (f \cdot g)(-3) \)
   d. \( \left(\frac{f}{g}\right)(-3) \)

In Exercises 3–8, use the given tables for \( f \) and \( g \) to form a table of values for each function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 0 )</td>
<td>(-2)</td>
<td>( 0 )</td>
<td>(-1)</td>
</tr>
<tr>
<td>( 3 )</td>
<td>( 0 )</td>
<td>( 3 )</td>
<td>( 5 )</td>
</tr>
<tr>
<td>( 8 )</td>
<td>( 7 )</td>
<td>( 8 )</td>
<td>( 9 )</td>
</tr>
</tbody>
</table>

3. \( f + g \)
4. \( f - g \)
5. \( g - f \)
6. \( f \cdot g \)
7. \( g \cdot f \)
8. \( \frac{f}{g} \)

In Exercises 9–12, use \( f = \{(2, -3), (1, 5), (4, 7)\} \) and \( g = \{(-2, 4), (1, -1), (4, 6)\} \) to form a set of ordered pairs for each function.

9. \( f - g \)
10. \( f + g \)
11. \( \frac{f}{g} \)
12. \( f \cdot g \)

In Exercises 13–16, use the given graphs for \( f \) and \( g \) to graph each function. (Hint: You may want first to write each function as a set of ordered pairs.)

#### Objective 2 Form the Composition of Two Functions

In Exercises 23 and 24, evaluate each expression, given 
\[ f(x) = x^2 - 1 \text{ and } g(x) = 2x + 5. \]

13. \( f + g \)
14. \( f - g \)
15. \( f \cdot g \)
16. \( \frac{f}{g} \)

In Exercises 17–22, determine whether the functions \( f \) and \( g \) are equal. If \( f \neq g \), state the reason.

17. \( f(x) = x - 3 \quad 18. \quad f(x) = 1 - \frac{1}{x} \)
\[ g(x) = \frac{x^2 - 5x + 6}{x - 2} \quad g(x) = \frac{x - 1}{x} \]

19. \( f = \{(-1, 1), (0, 0), (1, 1), (2, 4)\} \)
\[ g(x) = x^2 \]

20. \( f = \{(-7, 7), (0, 0), (8, 8)\} \)
\[ g(x) = |x| \]

21. \( f(x) = x \quad 22. \quad f(x) = 5 \)
\[ g(x) = \frac{x^2 + 4}{x^2 + 1} \quad g(x) = \frac{5x^2 + 15}{x^2 + 3} \]

Use the given tables for \( f \) and \( g \) to complete the tables for \( f \circ g \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( g(x) )</th>
<th>( x )</th>
<th>( f \circ g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 3 )</td>
<td>( 5 )</td>
<td>( 5 )</td>
<td>( 9 )</td>
<td>( 3 )</td>
<td>( 9 )</td>
</tr>
<tr>
<td>( 4 )</td>
<td>( 2 )</td>
<td>( 8 )</td>
<td>( 2 )</td>
<td>( 7 )</td>
<td>( 9 )</td>
</tr>
<tr>
<td>( 1 )</td>
<td>( 0 )</td>
<td>( 0 )</td>
<td>( 5 )</td>
<td>( 7 )</td>
<td>( 9 )</td>
</tr>
<tr>
<td>( 8 )</td>
<td>( 4 )</td>
<td>( 6 )</td>
<td>( 8 )</td>
<td>( 8 )</td>
<td>( 9 )</td>
</tr>
</tbody>
</table>

Use \( f = \{(1, 3), (4, 5), (6, 2)\} \) and \( g = \{(3, 6), (5, 1), (2, 4)\} \) to form a set of ordered pairs for:

13. \( f \circ g \)
14. \( g \circ f \)

25. \( f \circ g \)
26. \( g \circ f \)
27. Use the given graphs of \( f \) and \( g \) to graph \( f \circ g \). (Hint: You may want first to write each function as a set of ordered pairs.)

In Exercises 28–33, determine \( f \circ g \) and \( g \circ f \).

28. \( f(x) = 3x - 4 \)  \hspace{1cm} 29. \( f(x) = x^2 - 5x + 3 \)
   \hspace{1cm} 30. \( f(x) = \sqrt{x} \)  \hspace{1cm} 31. \( f(x) = x + 8 \)
   \hspace{1cm} 32. \( f(x) = \frac{1}{x} \)  \hspace{1cm} 33. \( f(x) = \frac{1}{x+2} \)

30. \( f(x) = x^3 + 1 \)

In Exercises 34–37, determine \( f + g, f - g, f \cdot g, \frac{f}{g} \), and \( f \circ g \) for the given functions. State the domain of the resulting function.

34. \( f(x) = 2x^2 - x - 3 \)  \hspace{1cm} 35. \( f(x) = 6x^2 - x - 15 \)
   \hspace{1cm} 36. \( f(x) = \frac{x}{x+1} \)  \hspace{1cm} 37. \( f(x) = 5x - 7 \)

In Exercises 38 and 39, use the graphs of \( f \) and \( g \) to graph \( f + g \).

In Exercises 40 and 41, determine both \( (f \circ f^{-1})(x) \) and \( (f^{-1} \circ f)(x) \).

40. \( f(x) = 2x - 3 \)  \hspace{1cm} 41. \( f(x) = \frac{2x - 1}{3} \)

42. Use \( f(x) = 4x + 5 \) to determine
   a. \( f^{-1}(x) \)  \hspace{1cm}  b. \( (f \circ f^{-1})(x) \)
   c. \( (f^{-1} \circ f)(x) \)

In Exercises 43–46, express each function in terms of \( f \) and \( g \) to graph

45. \( h(x) = \sqrt{x^2 + 1} \)

46. \( h(x) = \frac{\sqrt{x}}{x^2 + 1} \)

50. \( h(x) = x^3 - 2 + \frac{1}{x^3 - 2} \)

Connecting Concepts to Applications

51. Profit as the Difference of Revenue Minus Cost The weekly revenue function for \( u \) units of a product sold is \( R(u) = 4u^2 - 3u \) dollars, and the cost function for \( u \) units is \( C(u) = 10u + 25 \). Assume 0 is the least number of units that can be produced and 100 is the greatest number that can be marketed. Find the profit function \( P \), and find \( P(10) \), the profit made by selling 10 units.

52. Determining a Formula for Total Cost and Average Cost

53. Determining a Formula for Total Cost and Average Cost
54. Determining the Product of a Price Function and a Demand Function. The number of items demanded by consumers is a function of the number of months that the product has been advertised. The price per item is varied each month as part of the marketing strategy. The number demanded during the mth month is \( N(m) = 36m - m^2 \), and the price per item during the mth month is \( P(m) = 5m + 45 \). The revenue for the mth month \( R(m) \) is the product of the price per item and the number of items demanded. Determine

a. \( N(7) \)

b. \( P(7) \)

c. \( R(7) \)

d. \( R(m) \)

55. Composing Cost and Production Functions. The number of sofas a factory can produce weekly is a function of the number of hours it operates. This function is \( R(a) \). Express the radius \( r \) of this circle as a function of the length \( s \) of a side of the square.

56. Composing Cost and Markup Functions. The weekly cost \( C \) of making \( d \) doses of a vaccine is \( C(d) = 0.30d + 400 \). The company charges 150% of cost to its wholesaler for this drug; that is, \( R(C) = 1.5C \). Evaluate and interpret the following.

a. \( C(10) \)

b. \( C(50) \)

c. \( (C + S)(10) \)

d. \( (C + S)(7) \)

e. \( (C + S)(40) \)

f. \( (C + S)(100) \)

57. Composing an Area and a Radius Function. A circular concrete pad was poured to serve as the base for a grain bin. This pad was inscribed in a square plot as shown.

The break-even value (the value producing a profit of $0)

b. Express the area \( A \) of this circle as a function of the radius \( r \).

c. Determine \( (A \circ r)(s) \) and interpret this result.

58. Composing an Area and a Width Function. A metal box with an open top can be formed by cutting squares of sides \( x \) cm from each corner of a square piece of sheet metal of width 44 cm, as shown.

a. Express the area \( A \) of the border that is removed as a function of the remaining width \( w \).

b. Express the remaining width \( w \) as a function of \( x \).

c. Determine \( (A \circ w)(x) \) and interpret this function.

d. Express the volume \( V \) of the box as a function of \( x \).

Group discussion questions

60. Discovery Question. For \( f(x) = \log x \) and \( g(x) = 10^x \), determine \( (f \circ g)(x) \). What can you observe from this result?

61. Discovery Question

a. Write the inverse of the function \( f = \{(1, 4), (2, 5), (3, 3), (4, 1), (5, 2)\} \). Does \( f(x) = f^{-1}(x) \) for each input value of \( x \)?

b. Complete the function \( f = \{(1, 7), (2, 6), (3, 5), (4, _), (5, _), (6, _), (7, _)\} \) so that \( f(x) = f^{-1}(x) \).

c. Write the inverse of \( f(x) = \frac{1}{x} \). Does \( f(x) = f^{-1}(x) \) for each input value of \( x \)?
d. Have each member of your group write the equation of another function so that \( f(x) = f^{-1}(x) \). Then examine the graphs of all these functions for similar characteristics.

62. Discovery Question

a. For \( f(x) = \frac{3x - 2}{7x - 3} \), determine \( f^{-1} \).

b. Compare \( f \) and \( f^{-1} \).

c. What does this imply about the graph of \( y = f(x) \)?

### 12.5 Cumulative Review

1. The additive identity is \( \underline{\text{\ }} \).

2. The multiplicative identity is \( \underline{\text{\ }} \).

3. The additive inverse of 6 is \( \underline{\text{\ }} \).

4. The multiplicative inverse of 6 is \( \underline{\text{\ }} \).

5. The property that justifies rewriting \( 7x(3x - 2) + 5(3x - 2) \) as \( (7x + 5)(3x - 2) \) is the \( \underline{\text{\ }} \) of multiplication over addition.

### Section 12.6 Sequences, Series, and Summation Notation

**Objectives:**

1. Calculate the terms of arithmetic and geometric sequences.

2. Use summation notation and evaluate the series associated with a finite sequence.

3. Evaluate an infinite geometric series.

Objects and natural phenomena often form regular and interesting patterns. The study of these objects and phenomena generally involves data collected sequentially or in a systematic manner. Here are some examples of occurrences of sequences:

- Business: To enumerate the payments necessary to repay a loan
- Biology: To describe the growth pattern of a living organism
- Calculator design: To specify the sequence of terms used to calculate functions such as the exponential function \( e^x \) and the trigonometric function \( \cos x \)
- Calculus: To give the areas of a sequence of rectangles used to approximate the area of a region

We first examined arithmetic sequences in Section 2.1 and geometric sequences in Section 11.1. We now give a more formal definition of a sequence. A **sequence** is a function whose domain is a set of consecutive natural numbers. For example, the sequence 2, 5, 8, 11, 14, 17 can be viewed as the function \( \{(1, 2), (2, 5), (3, 8), (4, 11), (5, 14), (6, 17)\} \) with input values \( \{1, 2, 3, 4, 5, 6\} \) and output values \( \{2, 5, 8, 11, 14, 17\} \).

**Alternative Notations for the Sequence 2, 5, 8, 11, 14, 17**

<table>
<thead>
<tr>
<th>Mapping Notation</th>
<th>Function Notation</th>
<th>Subscript Notation</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \rightarrow ) 2</td>
<td>( f(1) = 2 )</td>
<td>( a_1 = 2 )</td>
</tr>
<tr>
<td>2 ( \rightarrow ) 5</td>
<td>( f(2) = 5 )</td>
<td>( a_2 = 5 )</td>
</tr>
<tr>
<td>3 ( \rightarrow ) 8</td>
<td>( f(3) = 8 )</td>
<td>( a_3 = 8 )</td>
</tr>
<tr>
<td>4 ( \rightarrow ) 11</td>
<td>( f(4) = 11 )</td>
<td>( a_4 = 11 )</td>
</tr>
<tr>
<td>5 ( \rightarrow ) 14</td>
<td>( f(5) = 14 )</td>
<td>( a_5 = 14 )</td>
</tr>
<tr>
<td>6 ( \rightarrow ) 17</td>
<td>( f(6) = 17 )</td>
<td>( a_6 = 17 )</td>
</tr>
</tbody>
</table>

A **finite sequence** has a last term. A sequence that continues without end is called an **infinite sequence**. A finite sequence with \( n \) terms can be denoted by \( a_1, a_2, \ldots, a_n \) and an infinite sequence can be denoted by \( a_1, a_2, \ldots, a_n, \ldots \). The notation consisting of three dots, which is used to represent the sequence \( a_1, a_2, \ldots, a_n \), is called ellipsis notation. This notation is used to indicate that terms in the sequence are missing in the listing but the pattern shown is continued.
1. Calculate the Terms of Arithmetic and Geometric Sequences

We now review the definitions and descriptions of arithmetic and geometric sequences.

### Arithmetic and Geometric Sequences

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Verbally</th>
<th>Numerical Example</th>
<th>Graphical Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Arithmetic</strong></td>
<td>$a_n - a_{n-1} = d$ or $a_n = a_{n-1} + d$</td>
<td>$-3, -1, 1, 3, 5, 7$</td>
<td><img src="image" alt="Graph of Arithmetic Sequence" /></td>
</tr>
<tr>
<td><strong>Geometric</strong></td>
<td>$\frac{a_n}{a_{n-1}} = r$ or $a_n = ra_{n-1}$</td>
<td>$\frac{1}{4}, \frac{1}{2}, 1, 2, 4, 8$</td>
<td><img src="image" alt="Graph of Geometric Sequence" /></td>
</tr>
</tbody>
</table>

### Example 1  Identifying Arithmetic and Geometric Sequences

Determine whether each sequence is arithmetic, geometric, both, or neither. If the sequence is arithmetic, write the common difference $d$. If the sequence is geometric, write the common ratio $r$.

**Solution**

(a) $17, 14, 11, 8, 5, \ldots$

Arithmetic sequence with $d = -3$

- $14 - 17 = -3$; $11 - 14 = -3$
- $8 - 11 = -3$; $5 - 8 = -3$

Not a geometric sequence

(b) $3, -6, 12, -24, 48, \ldots$

Not an arithmetic sequence

- $-6 - 3 = -9$
- $12 - (-6) = 18$
- $-9 \neq 18$

Geometric sequence with $r = -2$

- $\frac{6}{3} = -2$; $\frac{12}{-6} = -2$
- $\frac{-24}{12} = -2$; $\frac{48}{-24} = -2$
- $r = -2$

(c) $3, 3, 3, 3, \ldots$

Arithmetic sequence with $d = 0$

- $3 - 3 = 0$

Geometric sequence with $r = 1$

- $\frac{3}{3} = 1$
Since \( a_n \) can represent any term of the sequence, it is called the **general term**. The general term of a sequence sometimes is defined in terms of one or more of the preceding terms. A sequence defined in this manner is said to be **defined recursively**. Examples of recursive definitions for parts (a) and (b) of Example 1 are

\[(a) \quad a_n = a_{n-1} - 3 \quad \text{with} \quad a_1 = 17 \quad 17, 14, 11, 8, 5, \ldots\]
\[(b) \quad a_n = -2a_{n-1} \quad \text{with} \quad a_1 = 3 \quad 3, -6, 12, -24, 48, \ldots\]

Because recursive definitions relate terms to preceding terms, there must be an initial term or condition given so that the sequence can get started. This initial or “seed” value is frequently used by computer programmers to construct looping structures within a program. Example 2 reexamines the sequence in Example 1(d). This recursive definition requires two initial conditions.

**Self-Check 1**

Complete the finite sequence 3, 6, _____, _____, _____ so that the sequence will be

a. An arithmetic sequence
b. A geometric sequence

**Example 2 Using a Recursive Definition**

Write the first five terms of the sequence defined recursively by the formulas \( a_1 = 1 \), \( a_2 = 1 \), and \( a_n = a_{n-2} + a_{n-1} \).

**Solution**

\[
\begin{align*}
a_1 &= 1 & \text{The first two terms are given; for } n = 3, a_n \text{ is } a_1, a_{n-2} \text{ is } a_1, \text{ and } a_{n-1} \text{ is } a_2. \\
a_2 &= 1 \\
a_3 &= a_1 + a_2 = 1 + 1 & \text{Substitute 1 for } a_1 \text{ and 1 for } a_2. \\
a_3 &= 2 \\
a_4 &= a_2 + a_3 = 1 + 2 & \text{For } n = 4, a_n \text{ is } a_4, a_{n-2} \text{ is } a_2, \text{ and } a_{n-1} \text{ is } a_3. \\
a_4 &= 3 \\
a_5 &= a_3 + a_4 = 2 + 3 & \text{Substitute 1 for } a_2 \text{ and 2 for } a_3. \\
a_5 &= 5
\end{align*}
\]

**Answer:** 1, 1, 2, 3, 5

**Self-Check 2**

Use the recursive definition \( a_1 = 1, a_2 = 1, \) and \( a_n = a_{n-2} + a_{n-1} \) to write the next four terms of 1, 1, 2, 3, 5, _____, _____, _____, _____.
A spreadsheet is a very good tool to use to investigate sequences. The ability to define one cell in terms of others makes the use of recursive definitions very powerful. Graphing calculators also have a more limited capability using a seq feature to create sequences.

Recursive formulas are used extensively in computer science and in mathematics. However, these formulas do require us to compute a sequence term by term. For example, to compute the 100th term of an arithmetic sequence, we first need the 99th term.

Can I Determine the $n$th Term of an Arithmetic or Geometric Sequence Without Listing All of the Preceding Terms?

Yes, we will now develop formulas for these terms.

<table>
<thead>
<tr>
<th>Arithmetic Sequence</th>
<th>Geometric Sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_n = a_{n-1} + d$</td>
<td>$a_n = r a_{n-1}$</td>
</tr>
<tr>
<td>$a_1$</td>
<td>$a_1$</td>
</tr>
<tr>
<td>$a_2 = a_1 + d$</td>
<td>$a_2 = a_1 r$</td>
</tr>
<tr>
<td>$a_3 = a_2 + d = (a_1 + d) + d = a_1 + 2d$</td>
<td>$a_3 = a_2 r = (a_1 r) r = a_1 r^2$</td>
</tr>
<tr>
<td>$a_4 = a_3 + d = (a_1 + 2d) + d = a_1 + 3d$</td>
<td>$a_4 = a_3 r = (a_1 r^2) r = a_1 r^3$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\vdots$</td>
</tr>
<tr>
<td>$a_n = a_{n-1} + d = a_1 + (n - 1)d$</td>
<td>$a_n = a_{n-1} r = a_1 r^{n-1}$</td>
</tr>
<tr>
<td>$a_n = a_1 + (n - 1)d$</td>
<td>$a_n = a_1 r^{n-1}$</td>
</tr>
</tbody>
</table>

### Example 3 Calculating $a_n$ for an Arithmetic and a Geometric Sequence

Use the formulas for $a_n$ for an arithmetic sequence and a geometric sequence to calculate $a_{10}$ for each sequence.

#### (a) Determine $a_{10}$ for an arithmetic sequence with $a_1 = 8$ and $d = 7$.

**Solution**

Substitute the given values into the formula for $a_n$ for an arithmetic sequence.

- $a_n = a_1 + (n - 1)d$
- $a_{10} = 8 + (10 - 1)7$
- $a_{10} = 8 + 9(7)$
- $a_{10} = 8 + 63$
- $a_{10} = 71$

#### (b) Determine $a_{10}$ for a geometric sequence with $a_1 = 8$ and $r = 3$.

**Solution**

Substitute the given values into the formula for $a_n$ for a geometric sequence.

- $a_n = a_1 r^{n-1}$
- $a_{10} = 8(3)^{10-1}$
- $a_{10} = 8(3)^9$
- $a_{10} = 8(19,683)$
- $a_{10} = 157,464$

#### (c) Determine $a_{10}$ for a geometric sequence with $a_1 = 5$ and $r = -2$.

**Solution**

Because $r$ is negative, the terms in this geometric sequence will alternate in sign. All even-numbered terms will be negative.

- $a_n = a_1 r^{n-1}$
- $a_{10} = 5(-2)^{10-1}$
- $a_{10} = 5(-2)^9$
- $a_{10} = 5(-512)$
- $a_{10} = -2,560$

### Self-Check 3

Use the formulas $a_n = a_1 + (n - 1)d$ and $a_n = a_1 r^{n-1}$ to determine

- **a.** $a_2$ for an arithmetic sequence with $a_1 = 4$ and $d = 5$.
- **b.** $a_9$ for a geometric sequence with $a_1 = 4$ and $r = 5$. 

---

**CONFIRMING PAGES**
The formula $a_n = a_1 + (n - 1)d$ relates $a_n$, $a_1$, $n$, and $d$. Therefore, we can find any of these four variables when the other three are known. A similar statement can be made for the formula $a_n = a_1r^{n-1}$. This is illustrated in Example 4.

**Example 4** Determining the Common Difference and the Common Ratio

Use the formulas for $a_n$ for an arithmetic sequence and a geometric sequence to calculate $d$ and $r$.

**Solution**

(a) Find $d$ in an arithmetic sequence with $a_1 = -87$ and $a_{57} = 529$.

$$a_n = a_1 + (n - 1)d$$

$$529 = -87 + (57 - 1)d$$

$$616 = 56d$$

$$d = 11$$

Then solve for $d$.

(b) Find $r$ in a geometric sequence with $a_1 = 6$ and $a_3 = 24$.

$$a_n = a_1r^{n-1}$$

$$24 = (6)(r)^3 - 1$$

$$4 = r^2$$

$$r = 2 \text{ or } r = -2$$

Check: The geometric sequences $6, 12, 24, \ldots$ and $6, -12, 24, -48, \ldots$ both satisfy the given conditions.

**Self-Check 4**

Use the formulas $a_n = a_1 + (n - 1)d$ and $a_n = a_1r^{n-1}$ to determine

a. $d$ for an arithmetic sequence with $a_1 = 10$ and $a_{61} = 250$.

b. $r$ for a geometric sequence with $a_1 = 5$ and $a_4 = 1,080$.

Example 5 illustrates how to use the formulas for $a_n$ to calculate the number of terms in an arithmetic sequence and a geometric sequence.

**Example 5** Determining the Number of Terms in a Sequence

Use the formulas for $a_n$ for an arithmetic sequence and a geometric sequence to calculate the number of terms in each sequence.

**Solution**

(a) $-20, -13, -6, \ldots, 281$

$$d = -13 - (-20) = 7$$

$$a_n = a_1 + (n - 1)d$$

$$281 = -20 + (n - 1)7$$

$$301 = 7(n - 1)$$

$$n - 1 = 43$$

$$n = 44$$

First note that this is an arithmetic sequence with a common difference $d = 7$ and a first term of $-20$.

Substitute $a_1 = -20$, $a_n = 281$, and $d = 7$ into the formula for $a_n$ for an arithmetic sequence.

Then solve for $n$.

This arithmetic sequence has 44 terms.
This geometric sequence has six terms.

\[ r = \frac{1.250}{3.125} = \frac{500}{1250} = 0.4 \]

First note that this is a geometric sequence with a common ratio of \( r = 0.4 \).

\[ a_n = a_1 \cdot r^{n-1} \]

\[ a_6 = (3.125) (0.4)^{6-1} \]

\[ (0.4)^{n-1} = 0.01024 \]

\[ \log 0.4^{n-1} = \log 0.01024 \]

\[ (n - 1) \log 0.4 = \log 0.01024 \]

\[ n - 1 = \frac{\log 0.01024}{\log 0.4} \]

\[ n - 1 = 5 \]

\[ n = 6 \]

This geometric sequence has six terms.

You can check this answer by writing the first six terms of this geometric sequence.

**Self-Check 5**

Use the formulas \( a_n = a_1 + (n - 1)d \) and \( a_n = a_1 \cdot r^{n-1} \) to determine

a. The number of terms in the arithmetic sequence with \( a_1 = -12, a_n = 288 \), and \( d = 4 \).

b. The number of terms in the geometric sequence with \( a_1 = \frac{32}{3.125}, a_n = \frac{625}{16} \), and \( r = \frac{5}{2} \).

2. Use Summation Notation and Evaluate the Series Associated with a Finite Sequence

The sum of the terms of a sequence is called a series. If \( a_1, a_2, \ldots, a_n \) is a finite sequence, then the indicated sum \( a_1 + a_2 + \cdots + a_n \) is the series associated with this sequence. For arithmetic and geometric series, there are formulas that serve as shortcuts to evaluating the series. Example 6 illustrates the meaning of a series. Then we examine the shortcuts.

**Example 6** Evaluating an Arithmetic Series and a Geometric Series

Find the value of the six-term series associated with these arithmetic and geometric sequences.

**Solution**

(a) \( a_n = 3n \)

\[ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \]

\[ = 3(1) + 3(2) + 3(3) + 3(4) + 3(5) + 3(6) \]

\[ = 3 + 6 + 9 + 12 + 15 + 18 \]

\[ = 63 \]

The series is the sum of the first six terms. Substitute the first six natural numbers into the formula \( a_n = 3n \) to determine the first six terms, and then add the terms of this arithmetic series.

(b) \( a_n = 3^n \)

\[ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 \]

\[ = 3^1 + 3^2 + 3^3 + 3^4 + 3^5 + 3^6 \]

\[ = 3 + 9 + 27 + 81 + 243 + 729 \]

\[ = 1,092 \]

Calculate each of the six terms of this geometric sequence and then determine the sum.

**Self-Check 6**

Find the value of the five-term series associated with the sequence defined by \( a_n = 4n + 3 \).
A convenient way of denoting a series is to use **summation notation**, in which the Greek letter Σ (sigma, which corresponds to the letter S for “sum”) indicates the summation.

Generally the index variable is denoted by \( i \), \( j \), or \( k \). The index variable is always replaced with successive integers from the initial value through the last value. Example 7 illustrates that the initial value can be a value other than 1. For example, in \( \sum_{i=5}^{8} a_i \), \( i \) is replaced with 5, 6, 7, and then 8 to yield \( \sum_{i=5}^{8} a_i = a_5 + a_6 + a_7 + a_8 \).

### Example 7  Using Summation Notation to Evaluate an Arithmetic Series and a Geometric Series

Evaluate each series.

(a) \( \sum_{i=1}^{4} 5i \)

(b) \( \sum_{k=3}^{7} 2^k \)

**Solution**

(a) \( \sum_{i=1}^{4} 5i = 5(1) + 5(2) + 5(3) + 5(4) \)

\[= 5 + 10 + 15 + 20\]

\[= 50\]

(b) \( \sum_{k=3}^{7} 2^k = 2^3 + 2^4 + 2^5 + 2^6 + 2^7 \)

\[= 8 + 16 + 32 + 64 + 128\]

\[= 248\]

**Self-Check 7**

Evaluate each series.

a. \( \sum_{i=1}^{5} 4i \)

b. \( \sum_{j=1}^{4} (j^2 - 2j) \)

c. \( \sum_{k=3}^{6} 5^k \)

For a series with many terms, such as \( \sum_{i=1}^{100} i \), it is useful to have a shortcut formula that allows us to calculate the sum without actually doing all the adding. We now develop formulas for an arithmetic series and a geometric series.
Development of a Formula for an Arithmetic Series

\[ S_n = a_1 + (a_1 + d) + \cdots + [a_1 + (n - 2)d] + [a_1 + (n - 1)d] \]
\[ S_n = [a_1 + (n - 1)d] + [a_1 + (n - 2)d] + \cdots + (a_1 + d) + a_1 \]

\[ 2S_n = [a_1 + a_1 + (n - 1)d] + [a_1 + a_1 + (n - 2)d] + \cdots + [a_1 + a_1 + (n - 1)d] + [a_1 + a_1 + (n - 1)d] \]
\[ 2S_n = (a_1 + a_n) + (a_1 + a_n) + \cdots + (a_1 + a_n) + (a_1 + a_n) \]
\[ 2S_n = n(a_1 + a_n) \]
\[ S_n = \frac{n(a_1 + a_n)}{2} \]
\[ S_n = \frac{n[2a_1 + (n - 1)d]}{2} \]

Development of a Formula for a Geometric Series

\[ S_n = a_1 + a_1r + a_1r^2 + \cdots + a_1r^{n-2} + a_1r^{n-1} \]
\[ rS_n = a_1r + a_1r^2 + a_1r^3 + \cdots + a_1r^{n-2} + a_1r^{n-1} + a_1r^n \]

\[ S_n - rS_n = a_1 + 0 + 0 + \cdots + 0 + 0 - a_1r^n \]
\[ S_n(1 - r) = a_1(1 - r^n) \]
\[ S_n = \frac{a_1(1 - r^n)}{1 - r} \] for \( r \neq 1 \)
\[ S_n = \frac{a_1 - a_1r^n}{1 - r} \]
\[ S_n = \frac{a_1 - ra_n}{1 - r} \]

To obtain the second equation, multiply both sides of the first equation by \( r \) and shift terms to the right to align similar terms. Subtract the second equation from the first equation.

Factor both sides.

Divide both members by \( 1 - r \). If \( r \neq 1 \),
\[ 1 - r \neq 0. \]

Substitute \( a_n \) for \( a_1r^{n-1} \) to obtain an alternative form of this formula.

### Formulas for Arithmetic and Geometric Series

\[ S_n = \sum_{i=1}^{n} a_i \]

<table>
<thead>
<tr>
<th>Arithmetic series</th>
<th>Geometric series</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_n = \frac{n}{2}(a_1 + a_n) )</td>
<td>( S_n = \frac{a_1(1 - r^n)}{1 - r} )</td>
</tr>
<tr>
<td>or ( S_n = \frac{n}{2}[2a_1 + (n - 1)d] )</td>
<td>or ( S_n = \frac{a_1 - ra_n}{1 - r} )</td>
</tr>
<tr>
<td>( S_4 = \sum_{i=1}^{4} 5i = 5 + 10 + 15 + 20 )</td>
<td>( S_5 = \sum_{k=1}^{5} 2^k = 2 + 4 + 8 + 16 + 32 )</td>
</tr>
<tr>
<td>( S_4 = \frac{4}{2}(5 + 20) )</td>
<td>( S_5 = \frac{2(1 - 2^5)}{1 - 2} )</td>
</tr>
<tr>
<td>( S_4 = 2(25) )</td>
<td>( S_5 = \frac{2(-31)}{-1} )</td>
</tr>
<tr>
<td>( S_4 = 50 )</td>
<td>( S_5 = 62 )</td>
</tr>
</tbody>
</table>
Example 8 examines an application for both arithmetic and geometric series.

**Example 8** Using Arithmetic and Geometric Series to Model Applications

(a) Rolls of carpet are stacked in a warehouse, with 20 rolls on the first level, 19 on the second level, and so on. The top level has only 1 roll. How many rolls are in this stack? (See the figure.)

(b) If you could arrange to be paid $1,000 at the end of January, $2,000 at the end of February, $4,000 at the end of March, and so on, what is the total amount you would be paid for the year?

**Solution**

(a) The numbers of rolls on the various levels form an arithmetic sequence with \(a_1 = 20\), \(n = 20\), and \(a_{20} = 1\).

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Numerically</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_n = \frac{n(a_1 + a_n)}{2})</td>
<td>(S_n = a_1 + a_2 + \cdots + a_n)</td>
</tr>
<tr>
<td>(S_{20} = \frac{20(20 + 1)}{2})</td>
<td>(S_{20} = 20 + 19 + 18 + \cdots + 3 + 2 + 1)</td>
</tr>
<tr>
<td>(S_{20} = 210)</td>
<td>(S_{20} = 210)</td>
</tr>
</tbody>
</table>

**Answer:** There are 210 rolls in this stack.

(b) The payments at the end of each month form a geometric sequence with \(a_1 = 1,000\), \(r = 2\), \(n = 12\).

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Numerically</th>
</tr>
</thead>
<tbody>
<tr>
<td>(S_n = \frac{a_1(1 - r^n)}{1 - r})</td>
<td>(S_n = a_1 + a_2 + \cdots + a_n)</td>
</tr>
<tr>
<td>(S_{12} = \frac{1,000(1 - 2^{12})}{1 - 2})</td>
<td>(S_{12} = 1,000 + 2,000 + 4,000 + 8,000 + 16,000 + 32,000)</td>
</tr>
<tr>
<td>(S_{12} = 1,000(1 - 4,096))</td>
<td>+ (64,000 + 128,000 + 256,000)</td>
</tr>
<tr>
<td>(S_{12} = \frac{1,000(1 - 4,096)}{-1})</td>
<td>+ (512,000 + 1,024,000)</td>
</tr>
<tr>
<td>(S_{12} = -4,095,000)</td>
<td>+ (2,048,000)</td>
</tr>
<tr>
<td>(S_{12} = -4,095,000)</td>
<td>(S_{12} = 4,095,000)</td>
</tr>
<tr>
<td>(S_{12} = 4,095,000)</td>
<td></td>
</tr>
</tbody>
</table>

**Answer:** The total amount for the year would be $4,095,000.

**Self-Check 8**

Use the formulas for arithmetic and geometric series to determine these sums.

\[ \sum_{i=1}^{100} i \quad \text{and} \quad \sum_{i=1}^{20} 2^i \]

Compare the answers obtained algebraically in Example 8 with the answers obtained numerically by actually adding the terms in the series. Which method do you prefer? Which method would you prefer if there were 1,000 terms?
3. Evaluate an Infinite Geometric Series

If \(|r| < 1\), then the absolute values of the terms of the geometric sequence are decreasing. For example, the geometric sequence \(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots\) is a decreasing, infinite geometric sequence with \(r = \frac{1}{2}\). We already have a formula for finding the sum of a finite geometric sequence. Is the infinite sum meaningful? Although we could never actually add an infinite number of terms, we can adopt a new meaning for this sum. If the sum approaches some limiting value \(S\) as \(n\) becomes large, then we will call this value the **infinite sum**. Symbolically, \(\sum_{i=1}^{\infty} a_i = S\) if the sequence \(\sum_{i=1}^{n} a_i\) approaches \(S\) as \(n\) increases.

A general formula for the infinite sum can be obtained by examining the formula for \(S_n\):

\[
S_n = \frac{a_1(1 - r^n)}{1 - r}
\]

If \(|r| < 1\), then \(|r|^n\) approaches 0 as \(n\) becomes larger. Thus \(S_n = \frac{a_1(1 - r^n)}{1 - r}\) approaches \(\frac{a_1(1 - 0)}{1 - r}\); that is, \(S_n\) approaches \(\frac{a_1}{1 - r}\). The infinite sum \(S\) is the limiting value, so \(S = \frac{a_1}{1 - r}\). If \(|r| \geq 1\), the terms of a geometric series do not approach 0 and \(\sum_{i=1}^{n} a_i\) does not approach any limit as \(n\) becomes large. In this case, we do not assign a value to \(\sum_{i=1}^{\infty} a_i\).

### Infinite Geometric Series \(S = \sum_{i=1}^{\infty} a_i\)

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Algebraic Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>If (</td>
<td>r</td>
</tr>
<tr>
<td>If (</td>
<td>r</td>
</tr>
</tbody>
</table>

\[S = \frac{a_1}{1 - r}\]

\[
S = \frac{0.3}{1 - 0.1}
\]

\[
S = \frac{0.3}{0.9}
\]

\[
S = \frac{1}{3}
\]
Example 9  Writing a Repeating Decimal in Fractional Form

Write 0.272727... as a fraction.

**Solution**

\[
0.272727... = 0.27 + 0.0027 + 0.000027 + \cdots = 27(0.01) + 27(0.01)^2 + 27(0.01)^3 + \cdots = \sum_{i=1}^{\infty} 27(0.01)^i
\]

\[
S = \frac{a_1}{1 - r} = \frac{0.27}{1 - 0.01} = \frac{0.27}{0.99} = \frac{3}{11}
\]

**Answer:** 0.272727... = \frac{3}{11}

This series is an infinite geometric series with \(a_1 = 0.27\) and \(r = 0.01\).

This is the formula for the sum of an infinite geometric series.

Substitute 0.27 for \(a_1\) and 0.01 for \(r\).

Simplify and express \(S\) in fractional form.

You can check this answer by dividing 3 by 11.

Self-Check 9

Write 0.545454... as a fraction.

Self-Check Answers

1. a. 3, 6, 9, 12, 15  
   b. 3, 6, 12, 24, 48  
2. a, 1, 2, 3, 5, 8, 13, 21, 34  
   b, 1, 2, 3, 6, 10, 15  
3. a. \(a_{21} = 104\)  
   b. \(a_9 = 1,562,500\)  
4. a. \(d = 3\)  
   b. \(r = 6\)  
5. a. \(n = 76\)  
   b. \(n = 10\)  
6. 75  
7. a. 60  
   b. 10  
   c. 19,500  
8. a. 5,050  
   b. 2,097,150  
9. 0.545454... = \frac{6}{11}  
10. 12.6  

Using the Language and Symbolism of Mathematics

1. A sequence is a function whose domain is a set of consecutive _________ numbers.
2. A sequence that has a last term is called a _________ sequence.
3. A sequence that continues without end is called an _________ sequence.
4. A sequence with \(a_n = a_{n-1} + d\) is an _________ sequence.
5. A sequence with \(a_n = ra_{n-1}\) is a _________ sequence.
6. The sequence 1, 1, 2, 3, 5, 8, 13, ... is an example of a _________ sequence.
7. Since \(a_n\) can represent any term of a sequence, it is called the _________ term.
8. A sequence where the general term is defined in terms of one or more of the preceding terms is said to be defined _________.
9. The formula for \(a_n\) for an arithmetic sequence is \(a_n = \) _________.
10. The formula for \(a_n\) for a geometric sequence is \(a_n = \) _________.
11. In the _________ notation \(\sum_{i=1}^{n} a_i\), the index variable is _________.
    The initial value of the index variable in this example is _________, and the last value of the index variable is _________.
12. The formula for an arithmetic series is \(S_n = \) _________.
13. The formula for a geometric series is \(S_n = \) _________.
14. The formula for an infinite geometric series is \(S = \) _________ if \(|r| < 1\).
12.6 Quick Review

1. Write an algebraic expression for “$a$ sub 5 equals 23.”
2. Write an algebraic expression for “$a$ sub $n$ equals four $n$ plus eight.”

Determine the first five terms of each sequence.
3. $a_n = 3n + 5$
4. $a_n = n^2 - 5n + 8$
5. $a_n = 5 \cdot 2^{n-1}$

12.6 Exercises

Objective 1 Calculate the Terms of Arithmetic and Geometric Sequences

In Exercises 1–4, determine whether each sequence is arithmetic, geometric, both, or neither. If the sequence is arithmetic, write the common difference $d$. If the sequence is geometric, write the common ratio $r$.

1. a. 1, 6, 36, 216, 1,296, 7,776
   b. 1, 6, 11, 16, 21, 26
   c. 2, 3, 5, 8, 12, 13
   d. 6, 6, 6, 6, 6
2. a. 144, 124, 104, 84, 64, 44
   b. 144, 72, 36, 18, 9, 4.5
   c. 9, −9, −9, −9, −9
   d. 9, −8, 7, −6, 5, −4
3. a. $a_n = 3n - 5$
   b. $a_n = n^2$
   c. $a_n = 4^n$
   d. $a_n = 7$
4. a. $a_1 = 1$ and $a_n = 5a_{n-1}$ for $n > 1$
   b. $a_1 = 2$ and $a_n = (a_{n-1})^2$ for $n > 1$
   c. $a_1 = 3$ and $a_n = a_{n-1} + 5$ for $n > 1$
   d. $a_1 = 4$ and $a_n = 4$ for $n > 1$

In Exercises 5 and 6, write the first six terms of each sequence.

5. a. $a_n = 5n + 3$
   b. $a_n = n^2 - n + 1$
   c. $a_n = 16 \left( \frac{1}{2} \right)^n$
   d. $a_n = 5(-2)^n$
6. a. $a_1 = 4$ and $a_n = a_{n-1} + 5$ for $n > 1$
   b. $a_1 = 4$ and $a_n = 5a_{n-1}$ for $n > 1$
   c. $a_1 = 4$ and $a_n = -a_{n-1}$ for $n > 1$
   d. $a_1 = 1$, $a_2 = 2$, and $a_n = 2a_{n-2} + a_{n-1}$ for $n > 2$

In Exercises 7 and 8, write the first six terms of each arithmetic sequence.

7. a. 18, 14, ____, ____, ____, ____
   b. 18, ____, 14, ____, ____, ____
   c. $a_1 = 7$, $d = -2$
   d. $a_1 = -9$, $d = 3$
8. a. $a_1 = -8$, $a_6 = 12$
   b. $a_n = 4n - 1$
   c. $a_n = 11 - 2n$
   d. $a_1 = 1.2$ and $a_n = a_{n-1} + 0.4$ for $n > 1$

In Exercises 13–20, use the information given for the arithmetic sequence to find the quantities indicated.

13. 18, 14, 10, ____, −62; $n =$
14. 48, 55, ____, 496; $n =$
15. $a_{44} = 216$, $d = 12$, $a_1 =$
16. $a_{113} = -109$, $d = -2$, $a_1 =$
17. $a_{11} = 4$, $a_{31} = 14$, $d =$
18. $a_{47} = 23$, $a_{62} = 28$, $d =$
19. \(a_1 = 5, \quad a_n = 14, \quad d = \frac{1}{5}, \quad n = \) 
20. \(a_1 = -12, \quad a_n = 6, \quad d = \frac{1}{2}, \quad n = \) 

In Exercises 21–28, use the information given for the geometric sequences to find the quantities indicated.

21. \(243, 81, \ldots; \quad \frac{1}{3}; \quad n = \) 
22. \(1024, 512, \ldots; \quad 1; \quad n = \) 
23. \(a_5 = 24, \quad r = 2, \quad a_1 = \) 
24. \(a_6 = 64, \quad r = 4, \quad a_1 = \) 
25. \(a_9 = 32, \quad a_{11} = 288, \quad r = \) 
26. \(a_{25} = 17, \quad a_{37} = 425, \quad r = \) 
27. \(a_n = 5a_{n-1}, \quad a_1 = \frac{1}{3}, \quad a_9 = \) 
28. \(a_n = 0.1a_{n-1}, \quad a_1 = 7000, \quad a_8 = \) 

**Objective 2 Use Summation Notation and Evaluate the Series Associated with a Finite Sequence**

In Exercises 29 and 30, write the terms of each series and then add these terms.

29. a. \(\sum_{i=1}^{6} (2i + 3)\)  
   b. \(\sum_{i=1}^{5} (i^2 - 1)\)  
   c. \(\sum_{j=1}^{6} 2^j\)  
   d. \(\sum_{k=1}^{10} 5\)  
30. a. \(\sum_{j=1}^{7} \frac{k}{3} + 3\)  
   b. \(\sum_{j=1}^{8} (j^2 + 1)\)  
   c. \(\sum_{j=4}^{7} 3^j\)  
   d. \(\sum_{k=10}^{16} 7\)  

In Exercises 31–40, use the information given to evaluate each arithmetic series.

31. \(a_1 = 2, \quad a_{40} = 80, \quad S_{40} = \) 
32. \(a_1 = 3, \quad a_{31} = 153, \quad S_{31} = \) 
33. \(a_1 = \frac{1}{2}, \quad a_{12} = \frac{1}{3}, \quad S_{12} = \) 
34. \(a_1 = 0.36, \quad a_{18} = 0.64, \quad S_{18} = \) 
35. \(a_1 = 10, \quad d = 4, \quad S_{66} = \) 
36. \(a_1 = 11, \quad d = -3, \quad S_{11} = \) 
37. \(\sum_{i=1}^{61} (2i + 3)\)  
38. \(\sum_{k=1}^{47} (3k - 2)\)  
39. \(\sum_{k=1}^{24} \frac{k}{5} + 3\)  
40. \(\sum_{j=1}^{40} \frac{j - 5}{3}\)  

In Exercises 41–52, use the information given to evaluate each geometric series.

41. \(a_1 = 3, \quad r = 2, \quad S_7 = \) 
42. \(a_1 = 2, \quad r = 3, \quad S_6 = \) 
43. \(a_1 = 0.2, \quad r = 0.1, \quad S_5 = \) 
44. \(a_1 = 0.5, \quad r = 0.1, \quad S_6 = \) 
45. \(a_1 = 48, \quad r = -\frac{1}{2}, \quad S_8 = \) 
46. \(a_1 = 729, \quad r = \frac{1}{3}, \quad S_7 = \) 
47. \(a_1 = 1, \quad a_n = 3.71293, \quad r = 1.3, \quad S_n = \) 
48. \(64 + \frac{32 + \cdots + \frac{1}{8}}{9}\)  
49. \(\sum_{i=1}^{7} 8(0.1)^i\)  
50. \(\sum_{k=1}^{6} 7(0.1)^k\)  
51. \(a_1 = 40, \quad a_n = \frac{1}{5}a_{n-1} \quad \text{for} \quad n > 1, \quad S_6 = \) 
52. \(a_1 = 81, \quad a_n = \frac{1}{3}a_{n-1} \quad \text{for} \quad n > 1, \quad S_6 = \) 

**Objective 3 Evaluate an Infinite Geometric Series**

In Exercises 53–64, use the information given to evaluate the infinite geometric series.

53. \(a_1 = 5, \quad r = \frac{2}{3}\)  
54. \(a_1 = 16, \quad r = \frac{1}{5}\)  
55. \(a_1 = 14, \quad r = -\frac{3}{4}\)  
56. \(a_1 = 7, \quad r = -\frac{2}{5}\)  
57. \(a_1 = 0.12, \quad r = \frac{1}{100}\)  
58. \(a_1 = 0.9, \quad r = \frac{1}{10}\)  
59. \(\sum_{k=1}^{\infty} \left(\frac{4}{9}\right)^k\)  
60. \(\sum_{j=1}^{\infty} \left(\frac{3}{7}\right)^j\)  
61. \(6 - 4 + \frac{8}{3} - \frac{16}{9} + \cdots\)  
62. \(16 - 12 + 9 - 6.75 + \cdots\)  
63. \(a_1 = 24, \quad a_n = \frac{3}{8}a_{n-1} \quad \text{for} \quad n > 1\)  
64. \(a_1 = -36, \quad a_n = \left(-\frac{5}{9}\right) a_{n-1} \quad \text{for} \quad n > 1\)  

In Exercises 65 and 66, write each repeating decimal as a fraction.

65. a. \(0.444\ldots\)  
   b. \(0.212121\ldots\)  
   c. \(0.409409409\ldots\)  
   d. \(2.5555\ldots\)  
66. a. \(0.555\ldots\)  
   b. \(0.363636\ldots\)  
   c. \(0.495495495\ldots\)  
   d. \(8.3333\ldots\)

**Skill and Concept Development**

In Exercises 67–72, use the information given for the arithmetic sequences to find the quantities indicated.

67. \(a_{77} = 19, \quad d = -11, \quad S_{77} = \) 
68. \(S_n = 240, \quad a_1 = 4, \quad a_n = 16, \quad n = \) 
69. \(S_{30} = 1,560, \quad a_{30} = 93, \quad a_1 = \) 
70. \(S_{17} = 527, \quad a_1 = 15, \quad a_{17} = \) 
71. \(S_{40} = 680, \quad a_1 = 11, \quad d = \) 
72. \(S_{25} = 60, \quad a_1 = 17, \quad d = \)
In Exercises 73–78, use the information given for the geometric sequences to find the quantities indicated.

73. \( r = 2, S_{10} = 6,138, a_1 = \) _____

74. \( a_1 = 12, r = 5, S_n = 46,872, n = \) _____

75. \( a_1 = 6, a_n = 24,576, S_n = 32,766, r = \) _____

76. \( \sum_{i=1}^{\infty} a_i = 27, a_1 = 12, r = \) _____

77. \( \sum_{k=1}^{\infty} a_k = 21, r = \frac{2}{9}, a_1 = \) _____

78. \( \sum_{k=1}^{\infty} a_i = 2, r = \frac{6}{7}, a_1 = \) _____

Connecting Concepts to Applications

79. Stacks of Logs Logs are stacked so that each layer after the first has 1 less log than the previous layer. If the bottom layer has 24 logs and the top layer has 8 logs, how many logs are in the stack? (See the figure.)

80. Rolls of Insulation Rolls of insulation are stacked so that each layer after the first has 8 fewer rolls than the previous layer. How many layers will a lumberyard need to use in order to stack 120 rolls if 40 rolls are placed on the bottom layer? (See the figure.)

81. Increased Productivity The productivity gain from installing a new robot welder on a machinery assembly line is estimated to be $4,000 the first month of operation, $4,500 the second month, and $5,000 the third month. If this trend continues, what will be the total productivity gain for the first 12 months of operation?

82. Seats in a Theater A theater has 20 rows of seats, with 100 seats in the back row. Each row has 2 fewer seats than the row immediately behind it. How many seats are in the theater?

83. Chain Letter A chain-letter scam requires that each participant persuade four other people to participate. If one person starts this venture as a first-generation participant, determine how many people will have been involved by the time the eighth generation has signed on but not yet contacted anyone.

84. Bouncing Ball A ball dropped from a height of 36 m rebounds to six-tenths its previous height on each bounce. How far has it traveled when it reaches the apex of its eighth bounce? Give your answer to the nearest tenth of a meter. (Hint: You may wish to consider the distances going up separately from the distances going down.)

85. Vacuum Pump With each cycle a vacuum pump removes one-third of the air in a glass vessel. What percent of the air has been removed after eight cycles?

86. Arc of a Swing A child’s swing moves through a 3-m arc. On each swing, it travels only two-thirds the distance it traveled on the previous arc. How far does the swing travel before coming to rest?

87. Multiplier Effect City planners estimate that a new manufacturing plant located in their area will contribute $600,000 in salaries to the local economy. They estimate that those who earn the salaries will spend three-fourths of this money within the community. The merchants, service providers, and others who receive
Chapter 12 A Preview of College Algebra

this $450,000 from the salary earners in turn will spend three-fourths of it in the community, and so on. Taking into account the multiplier effect, find the total amount of spending within the local economy that will be generated by the salaries from this new plant.

Group discussion questions

88. Challenge Question If 20 people in a room shake hands with each other exactly once, how many handshakes will take place?

89. Discovery Question The trichotomy property of real numbers states that exactly one of the following three statements must be true in each case. Determine which statement is true.
   a. i. $0.333 \ldots < \frac{1}{3}$ ii. $0.333 \ldots = \frac{1}{3}$ iii. $0.333 \ldots > \frac{1}{3}$
   b. i. $0.666 \ldots < \frac{2}{3}$ ii. $0.666 \ldots = \frac{2}{3}$ iii. $0.666 \ldots > \frac{2}{3}$
   c. i. $0.888 \ldots < \frac{8}{9}$ ii. $0.888 \ldots = \frac{8}{9}$ iii. $0.888 \ldots > \frac{8}{9}$
   d. What is the meaning of the three dots in $0.999 \ldots$?
   e. Which of the following is true?
      i. $0.999 \ldots < 1$ ii. $0.999 \ldots = 1$ iii. $0.999 \ldots > 1$
   f. Are there any real numbers between $0.999 \ldots$ and $1$? If so, give an example.
   g. Multiply both sides of $\frac{1}{3} = 0.333 \ldots$ by $3$. What does this prove?
   h. Is the statement, “The number one-half on the number line can be represented by $\frac{1}{2}$, 50%, $0.5$, and $0.499 \ldots$” a true statement? If not, why is it false?

90. Risks and Choices One gimmick advertised in a newspaper was a “sure-fire” secret formula for becoming a millionaire. The secret discovered by those unwise enough to pay for it was as follows: “On the first day of the month save 1¢, on the second day save 2¢, on the third 4¢, etc.” How many days would be required to save a total of at least $1,000,000? At this rate, what amount would be saved on the last day?

12.6 Cumulative Review

1. Write 12,000 in scientific notation.
2. Write 0.0045 in scientific notation.
3. Write an inequality for the interval $(-3, 5)$.
4. Write an inequality for the interval $(-\infty, 2]$.
5. Write the interval notation for the inequality $2 \leq x \leq 7$.

Section 12.7 Conic Sections

Objective:
1. Graph the conic sections and write their equations in standard form.

What Is Meant by the Term Conic Sections?
Parabolas, circles, ellipses, and hyperbolas can be formed by cutting a cone or a pair of cones with a plane. Therefore these figures collectively are referred to as conic sections. Conic sections originally were studied from a geometric viewpoint. (See Fig. 12.7.1.)

Figure 12.7.1 Conic sections.
Two formulas that are useful for examining relationships between points are the distance formula (Section 10.4) and the midpoint formula. Note that the midpoint is found by taking the average of the $x$-coordinates and the average of the $y$-coordinates.

The distance and midpoint formulas are summarized in the following box.

### Distance and Midpoint Formulas

If $(x_1, y_1)$ and $(x_2, y_2)$ are two points, then:

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Numerical Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Distance formula:</strong> $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$</td>
<td>For $(-2, 7)$ and $(4, -1)$: Distance between the points: $d = \sqrt{(4 - (-2))^2 + (-1 - 7)^2}$ $d = \sqrt{36 + 64} = \sqrt{100}$ $d = 10$ Midpoint between the points: $(x, y) = \left(\frac{-2 + 4}{2}, \frac{-1 - 7}{2}\right) = \left(\frac{2}{2}, \frac{6}{2}\right)$ $(x, y) = (1, 3)$</td>
</tr>
<tr>
<td><strong>Midpoint formula:</strong> $(x, y) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$</td>
<td></td>
</tr>
</tbody>
</table>

### 1. Graph the Conic Sections and Write Their Equations in Standard Form

Using the distance formula, we now develop the equation of a circle with center $(h, k)$ and radius $r$. A circle is the set of all points in a plane that are a constant distance from a fixed point. (See Fig. 12.7.2.) The fixed point is called the center of the circle, and the distance from the center to the points on the circle is called the length of a radius. A diameter is a line segment from one point on a circle through the center to another point on the circle. The length of a diameter is twice the length of a radius.

The distance $r$ from any point $(x, y)$ on the circle to the center $(h, k)$ is given by

$$r = \sqrt{(x - h)^2 + (y - k)^2}$$
A Mathematical Note
The term diameter is formed by combining the Greek words dia meaning “across” and metros meaning to “measure.” Euclid used diameter as an appropriate name for the chord across a circle through the circle’s center.

Squaring both sides of this equation gives an equation satisfied by the points on the circle:

\[(x - h)^2 + (y - k)^2 = r^2\]

**Figure 12.7.2** To draw a circle, fix a loop of string with a tack and draw the circle as illustrated.

### Standard Form of the Equation of a Circle

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Graphically</th>
<th>Algebraic Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>The equation of a circle with center ((h, k)) and radius (r) is ((x - h)^2 + (y - k)^2 = r^2)</td>
<td><img src="image" alt="Graphical representation of a circle centered at ((h, k)) with radius (r)" /></td>
<td>The equation of a circle with center ((-1, 3)) and radius 5 is ((x - (-1))^2 + (y - 3)^2 = 5^2) [(x + 1)^2 + (y - 3)^2 = 25]</td>
</tr>
</tbody>
</table>

### Example 1 Determining the Equation of a Circle

Determine the equation of each graphed circle.

(a) \(h, k = (4, -3)\)

**Solution**

- Radius \(r = \sqrt{(6 - 4)^2 + [1 - 3 - (-3)]^2} = \sqrt{4} = 2\)
- Circle: \((x - h)^2 + (y - k)^2 = r^2\)
  \[(x - 4)^2 + (y + 3)^2 = 4\]

(b) \(h, k = (6, -3)\)

Determine the center by inspection, and use the distance formula to calculate the length of the radius. The segment \((4, -3)\) to \((6, -3)\) is a radius of this circle.

Substitute the center and radius into the standard form for the equation of a circle.
Center: $(h, k) = \left( \frac{1 + 9}{2}, \frac{1 + 1}{2} \right)$

Radius: $r = \sqrt{(9 - 5)^2 + (1 - 1)^2}$

Circle: $(x - h)^2 + (y - k)^2 = r^2$

Use the midpoint formula to calculate the center of the circle with a diameter from $(1, 1)$ to $(9, 1)$. Then use the center, one of the points on the circle, and the distance formula to calculate the length of the radius. Substitute the center and radius into the standard form for the equation of a circle.

**Self-Check 1**

A circle has a diameter from $(-3, 4)$ to $(3, -4)$.

a. Determine the center of this circle.

b. Determine the length of this diameter.

c. Determine the length of a radius.

d. Determine the equation of this circle.

The standard form of the circle $(x - 4)^2 + (y + 3)^2 = 4$ can be expanded to give the general form $x^2 + y^2 - 8x + 6y + 21 = 0$. If we are given the equation of a circle in general form, we can use the process of completing the square to rewrite the equation in standard form so that the center and radius will be obvious.

**Example 2  Writing the Equation of a Circle in Standard Form**

Determine the center and the radius of the circle defined by the equation $3x^2 + 3y^2 + 30x - 66y + 330 = 0$.

**Solution**

\[
\begin{align*}
3x^2 + 3y^2 + 30x - 66y + 330 &= 0 \\
x^2 + y^2 + 10x - 22y + 110 &= 0 \\
(x^2 + 10x) + (y^2 - 22y) &= -110 \\
(x^2 + 10x + 25) + (y^2 - 22y + 121) &= -110 + 25 + 121 \\
(x + 5)^2 + (y - 11)^2 &= 36 \\
[x - (-5)]^2 + (y - 11)^2 &= 6^2
\end{align*}
\]

**Answer:** The circle has center $(-5, 11)$ and radius 6.

**Self-Check 2**

Write $4x^2 + 4y^2 - 24x + 32y + 99 = 0$ in standard form and identify the center and the radius.

Equations of the form $x^2 + y^2 = c$, for $c > 0$, define circles centered at the origin. The following box also describes the cases where $c = 0$ or $c < 0$. 

---

12.7  Conic Sections  (12-79)  79
By the vertical line test, circles are not functions. Thus graphing calculators do not graph circles with $y$ as a function of $x$. Nonetheless, there are ways to get the graph of a circle to appear on a graphing calculator display. One way is to split a circle into two semicircles—an upper semicircle and a lower semicircle, both of which are functions. This is illustrated with the circle $x^2 + y^2 = 1$. Solving for $y$, we obtain $y = \pm \sqrt{1 - x^2}$. The upper semicircle is defined by $Y_1 = +\sqrt{1 - x^2}$, and the lower semicircle is defined by $Y_2 = -\sqrt{1 - x^2}$.

An ellipse is the set of all points in a plane, the sum of whose distances from two fixed points is constant. (See Fig. 12.7.3.) The two fixed points $F_1(-c, 0)$ and $F_2(c, 0)$ are called foci. The major axis of the ellipse passes through the foci. The minor axis is shorter than the major axis and is perpendicular to it at the center. The ends of the major axis are called the vertices, and the ends of the minor axis are called the covertices. If the ellipse is centered at the origin, then the equation of the ellipse as shown in Fig. 12.7.4 is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ where $a > b$.

**Equations of the Form $x^2 + y^2 = c$**

As shown in the figure,

1. $x^2 + y^2 = 4$ is a circle with center $(0, 0)$ and radius 2.
2. $x^2 + y^2 = 1$ is a circle with center $(0, 0)$ and radius 1.
3. $x^2 + y^2 = \frac{1}{4}$ is a circle with center $(0, 0)$ and radius $\frac{1}{2}$.
4. $x^2 + y^2 = 0$ is the degenerate case of a circle—a single point $(0, 0)$.
5. $x^2 + y^2 = -1$ has no real solutions and thus no points to graph. Both $x^2$ and $y^2$ are nonnegative, so $x^2 + y^2$ also must be nonnegative.

**Figure 12.7.3**  To draw an ellipse, loop a piece of string around two tacks and draw the ellipse as illustrated.

**Figure 12.7.4**  $d_1 + d_2 = 2a.$
We now use translations (Section 12.3) to examine ellipses that are not centered at the origin. The graph of \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) is identical to the graph of \( \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 \) except that it has been translated so that the center is at \((h, k)\) instead of \((0, 0)\).

### Standard Form of the Equation of an Ellipse

The equation of an ellipse with center \((h, k)\), major axis of length \(2a\), and minor axis of length \(2b\) is:

<table>
<thead>
<tr>
<th>Algebrically</th>
<th>Algebraic Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal major axis: ( \frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1 )</td>
<td>( \frac{(x - 2)^2}{49} + \frac{(y + 3)^2}{16} = 1 )</td>
</tr>
</tbody>
</table>

Example 3  Graphing an Ellipse Centered at the Origin

Graph \( \frac{x^2}{49} + \frac{y^2}{16} = 1 \).

**Solution**

Plot the \( x \)-intercepts \((-7, 0) \) and \((7, 0)\).
Plot the \( y \)-intercepts \((0, -4) \) and \((0, 4)\).

Then use the known shape to complete the ellipse.

From the given equation \( \frac{x^2}{49} + \frac{y^2}{16} = 1 \), this is an ellipse centered at the origin with \( a^2 = 49 \) and \( b^2 = 16 \). Thus \( a = 7 \), \( b = 4 \), and the intercepts are \((-7, 0), (7, 0), (0, -4), \) and \((0, 4)\).
Example 4  Writing the Equation of an Ellipse in Standard Form

Determine the equation of the graphed ellipse.

**Solution**

\[
(h, k) = \left(\frac{-1 + 7 + 1 + 1}{2}, \frac{2}{2}\right)
\]

\[
(h, k) = (3, 1)
\]

\[
2a = \sqrt{(7 - (-1))^2 + (1 - 1)^2}
\]

\[
2a = \sqrt{8^2 + 0^2}
\]

\[
2a = \sqrt{64}
\]

\[
a = 8
\]

\[
b = \sqrt{(3 - 3)^2 + (4 - (-2))^2}
\]

\[
b = \sqrt{0^2 + 6^2}
\]

\[
b = \sqrt{36}
\]

\[
b = 6
\]

\[
\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1
\]

The center of the ellipse is at the midpoint of the major axis.

The length of the major axis is \(2a\), and the length of the minor axis is \(2b\). Use the distance formula to determine these values.

The horizontal axis is the major axis; thus we use the standard form of an ellipse with a horizontal major axis.

Substitute \(a\), \(b\), and \(k\) into the standard form.

**Self-Check 4**

Write the equation \(9x^2 + 16y^2 - 72x - 96y + 144 = 0\) in standard form and graph the ellipse.
A hyperbola is the set of all points in a plane whose distances from two fixed points have a constant difference. (See Fig. 12.7.5.) The fixed points are the foci of the hyperbola. If the hyperbola is centered at the origin and opens horizontally as in Fig. 12.7.6, then the equation of the hyperbola is \( \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \). This hyperbola is asymptotic to the lines \( y = -\frac{b}{a}x \) and \( y = \frac{b}{a}x \). As \( |x| \) becomes larger, the hyperbola gets closer and closer to these lines. The asymptotes pass through the corners of the rectangle formed by \( (a, b), (-a, b), (-a, -b), \) and \( (a, -b) \). This rectangle, which is shown in Fig. 12.7.6, is called the fundamental rectangle and is used to sketch quickly the linear asymptotes.

\[ \begin{align*}
\text{Figure 12.7.5} & \quad |d_1 - d_2| = 2a. \\
\text{Figure 12.7.6} & \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.
\end{align*} \]

**Example 5** Graphing a Hyperbola Centered at the Origin

Graph \( \frac{x^2}{16} - \frac{y^2}{9} = 1. \)

**Solution**
Plot the \( x \)-intercepts \((-4, 0) \) and \((4, 0) \). Sketch the fundamental rectangle with corners \((4, 3), (-4, 3), (-4, -3), \) and \((4, -3) \). Draw the asymptotes through the corners of this rectangle. Then sketch the hyperbola through the \( x \)-intercepts that opens horizontally, using the asymptotes as guidelines.

From the given equation \( \frac{x^2}{16} - \frac{y^2}{9} = 1 \), this is a hyperbola centered at the origin with \( a^2 = 16 \) and \( b^2 = 9 \). Thus \( a = 4 \) and \( b = 3 \).

**Self-Check 5**
Graph \( \frac{x^2}{25} - \frac{y^2}{4} = 1. \)
We now use translations to examine hyperbolas whose center is \((h, k)\). The following box covers both hyperbolas that open horizontally and those that open vertically.

**Standard Form of the Equation of a Hyperbola**

The equation of a hyperbola with center \((h, k)\) is

<table>
<thead>
<tr>
<th>Algebraically</th>
<th>Algebraic Example</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Opening horizontally:</strong></td>
<td></td>
</tr>
</tbody>
</table>
| \[
\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1
\] | \[
\frac{(x + 1)^2}{16} - \frac{(y - 2)^2}{9} = 1
\] |

Vertices: \((h - a, k)\) and \((h + a, k)\).
The fundamental rectangle has base \(2a\) and height \(2b\).

<table>
<thead>
<tr>
<th>Opening vertically:</th>
<th></th>
</tr>
</thead>
</table>
| \[
\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1
\] | \[
\frac{(y - 2)^2}{9} - \frac{(x + 1)^2}{16} = 1
\] |

Vertices: \((h, k - a)\) and \((h, k + a)\).
The fundamental rectangle has base \(2b\) and height \(2a\).
We have already graphed parabolas in several sections of this book (including Section 8.4). For completeness of this introduction to conic sections, we now note that a parabola can be formed by cutting a cone with a plane. A **parabola** is the set of all points in a plane the same distance from a fixed line \(L\) (the **directrix**) and from a fixed point \(F\) (the **focus**).

See Fig. 12.7.7. Note that the axis of symmetry passes through the vertex, and it is perpendicular to the directrix. If the vertex of the parabola in Fig. 12.7.7 is at the origin, then the equation of this parabola can be written in the form

\[
\frac{y - k}{a^2} - \frac{(x - h)^2}{b^2} = 1
\]

The center of the hyperbola was determined by inspection.

Select the form for a hyperbola that opens vertically.

Use points on the fundamental rectangle to compute the values of \(a\) and \(b\).

Substitute \((-3, -5)\) for \((h, k)\) and 6 for \(a\) and 3 for \(b\) into the standard form for this hyperbola.

**Self-Check 6**

Write the equation of the hyperbola graphed in the figure.

We have already graphed parabolas in several sections of this book (including Section 8.4). For completeness of this introduction to conic sections, we now note that a parabola can be formed by cutting a cone with a plane. A **parabola** is the set of all points in a plane the same distance from a fixed line \(L\) (the **directrix**) and from a fixed point \(F\) (the **focus**). See Fig. 12.7.7. Note that the axis of symmetry passes through the vertex, and it is perpendicular to the directrix. If the vertex of the parabola in Fig. 12.7.7 is at the origin, then the equation of this parabola can be written in the form

\[
y = \frac{1}{4p}x^2.
\]

If the vertex is translated to the point \((h, k)\), then the equation becomes

\[
y - k = \frac{1}{4p}(x - h)^2.
\]

The equation \(y + 2 = \frac{1}{4}\left(\frac{1}{4}\right)(x - 3)^2\) can also be rewritten in the form

\[
y = (x - 3)^2 - 2.
\]

This form reveals the horizontal and vertical translations that we covered in Section 12.3.

**Example 6** **Writing the Equation of a Hyperbola in Standard Form**

Determine the equation of the hyperbola graphed in the figure.

**Solution**

\[(h, k) = (-3, -5)\]

\[
\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1
\]

\[
2a = \sqrt{[-3 - (-3)]^2 + [1 - (-11)]^2}
\]

\[
2a = \sqrt{0 + 12^2} = \sqrt{144}
\]

\[
a = 6
\]

\[
2b = \sqrt{(-6 - 0)^2 + [-5 - (-5)]^2}
\]

\[
2b = \sqrt{(-6)^2 + 0} = \sqrt{36}
\]

\[
b = 6
\]

\[
\frac{[y - (-5)]^2}{6^2} - \frac{[x - (-3)]^2}{3^2} = 1
\]

\[
\frac{(y + 5)^2}{36} - \frac{(x + 3)^2}{9} = 1
\]
Example 8 involves a circle and a parabola and uses the graphs of these conic sections to solve the corresponding system of equations and inequalities.

Example 7  Graphing a Parabola

Graph \( y = (x - 3)^2 - 2 \).

Solution

This parabola can be obtained by translating the graph of \( y = x^2 \) right 3 units and down 2 units. The vertex of this parabola is \((3, -2)\), and the \( y \)-intercept is \((0, 7)\).

Self-Check 7

Graph \( y = -(x + 2)^2 + 1 \).

Example 8  Solving a Nonlinear System of Equations and a Nonlinear System of Inequalities

Use graphs to solve.

(a) \( x^2 + y^2 = 25 \)  \hspace{1cm} (b) \( x^2 + y^2 \leq 25 \)
\[
y = x^2 - 5 \hspace{1cm} y \geq x^2 - 5
\]

Solution

(a) \( x^2 + y^2 = 25 \) is a circle centered at the origin with radius \( r = 5 \).
\( y = x^2 - 5 \) is a parabola opening upward with its vertex at \((0, -5)\).

\((-3, 4), (0, -5), \) and \((3, 4)\) all satisfy both equations.

Do all these points check?
The shaded points satisfy both inequalities.

**Self-Check 8**
Sketch the graph of the solution of \( \begin{cases} y \leq x + 2 \\ x^2 + y^2 \leq 4 \end{cases} \).

**Self-Check Answers**

1. a. (0, 0)
   
   b. 10
   
   c. 5
   
   d. \( x^2 + y^2 = 25 \)
2. \( (x - 3)^2 + (y - (-4))^2 = \left( \frac{1}{2} \right)^2 \) with center \((3, -4)\) and radius \( r = \frac{1}{2} \).
3.
4. \( \frac{(x - 4)^2}{16} + \frac{(y - 3)^2}{9} = 1 \)
5.
6. \( \frac{(y - 2)^2}{4} - \frac{(x - 3)^2}{25} = 1 \)

The points satisfying \( x^2 + y^2 \leq 25 \) lie on and inside the circle. Use \((0, 0)\) as a test value. The points satisfying \( y \leq x^2 - 5 \) lie on and above the parabola. Use \((0, 0)\) as a test value.
12.7 Using the Language and Symbolism of Mathematics

1. Parabolas, circles, ellipses, and hyperbolas are collectively referred to as __________ __________.
2. A __________ is the set of all points in a plane that are a constant distance from a fixed point.
3. An __________ is the set of all points in a plane the sum of whose distances from two fixed points is constant.
4. A __________ is the set of all points in a plane whose distances from two fixed points have a constant difference.
5. A __________ is the set of all points in a plane the same distance from a fixed line and a fixed point.

12.7 Quick Review

1. The vertex of the parabola defined by \( ax^2 + bx + c = 0 \) has an \( x \)-coordinate of __________ and a \( y \)-coordinate of __________.
2. The vertex of the parabola defined by \( f(x) = 2x^2 + 4x + 3 \) is __________.
3. Expand \( 9(x - 2)^2 + 4(y + 1)^2 - 36 \) and simplify the result.
4. Solve \( 2x^2 + 5x + 7 = 0 \) by completing the square.
5. Complete the square to rewrite the function \( f(x) = x^2 + 12x - 30 \) in the form \( f(x) = a(x - h)^2 + k \).

12.7 Exercises

Objective 1 Graph the Conic Sections and Write Their Equations in Standard Form

In Exercises 1–4, calculate the distance between each pair of points and the midpoint between these points.
1. \((-3, 2)\) and \((1, -1)\)
2. \((-6, 2)\) and \((6, 3)\)
3. \((0, 1)\) and \((1, 2)\)
4. \((a + 1, b)\) and \((a - 3, b + 3)\)

5. Calculate the length of the radius of a circle with center at \((0, 0)\) and with the point \((7, 24)\) on the circle.
6. Calculate the length of the diameter of a circle with endpoints on the circle at \((-8, 0)\) and \((15, 0)\).
7. A circle has a diameter with endpoints at \((-1, 5)\) and \((5, -3)\). Determine the center of this circle.

In Exercises 8–12, match each graph with the corresponding equation.
8. \( y = x^2 + 4 \)
9. \( x^2 + y^2 = 4 \)
10. \( \frac{x^2}{1} + \frac{y^2}{4} = 1 \)
11. \( \frac{x^2}{1} - \frac{y^2}{4} = 1 \)
12. \( y = x + 4 \)
In Exercises 13–17, write in standard form the equation of the circle satisfying the given conditions.
13. Center (0, 0), radius 10
14. Center (5, −4), radius 4
15. Center (2, 6), radius \( \sqrt{2} \)
16. Center (1, 1), radius 0.1
17. Center \( \left( 0, \frac{1}{2} \right) \), radius \( \frac{1}{2} \)

In Exercises 18–24, determine the center and the length of the radius of the circle defined by each equation.
18. \( x^2 + y^2 = 144 \)
19. \( (x + 5)^2 + (y − 4)^2 = 64 \)
20. \( (x − 4)^2 + (y + 5)^2 = 36 \)
21. \( x^2 + y^2 − 6x = 0 \)
22. \( x^2 + y^2 + 12y − 13 = 0 \)
23. \( x^2 + y^2 − 2x + 10y + 22 = 0 \)
24. \( 4x^2 + 4y^2 − 8x − 40y + 103 = 0 \)

In Exercises 25–28, determine the center and the lengths of the major and minor axes of the ellipse defined by each equation, and then graph the ellipse.
25. \( \frac{(x − 3)^2}{36} + \frac{(y + 2)^2}{16} = 1 \)
26. \( \frac{(x − 1)^2}{9} + \frac{(y + 2)^2}{49} = 1 \)
27. \( \frac{(y + 3)^2}{25} + \frac{(x − 4)^2}{16} = 1 \)
28. \( 36x^2 + 49y^2 = 1,764 \)

In Exercises 29–36, write in standard form the equation of the ellipse satisfying the given conditions.
29. Center (0, 0), x-intercepts (−9, 0) and (9, 0),
y-intercepts (0, −5) and (0, 5)
30. Center (0, 0), x-intercepts (−5, 0) and (5, 0),
y-intercepts (0, −3) and (0, 3)
31. Center (3, 4), horizontal major axis of length 4, vertical minor axis of length 2
32. Center (−5, 2), horizontal major axis of length 6, vertical minor axis of length 2
33. Center (6, −2), vertical major axis of length 10, horizontal minor axis of length 6
34. Center (−3, −4), vertical major axis of length 8, horizontal minor axis of length \( \frac{1}{2} \)
35. Center (−3, −4), \( a = 6, b = 2 \), major axis vertical
36. Center (2, 5), \( a = 8, b = 4 \), major axis horizontal

In Exercises 37–42, determine the center, the values of \( a \) and \( b \), and the direction in which the hyperbola opens, and then sketch the hyperbola.
37. \( \frac{x^2}{25} − \frac{y^2}{81} = 1 \)
38. \( \frac{x^2}{36} − \frac{y^2}{16} = 1 \)
39. \( 25x^2 = 100y^2 + 100 \)
40. \( \frac{(x − 4)^2}{49} − \frac{(y − 6)^2}{9} = 1 \)
41. \( \frac{(y − 6)^2}{9} − \frac{(x − 4)^2}{49} = 1 \)
42. \( 16x^2 + 96x = 9y^2 − 126y + 153 \)

In Exercises 43–46, write the standard form of the equation of the hyperbola satisfying the given conditions.
43. The center is (0, 0), the hyperbola opens vertically, and the fundamental rectangle has height 10 and width 8.
44. The center is (0, 0), the hyperbola opens vertically, and the fundamental rectangle has height 6 and width 12.
45. The hyperbola has vertices (−3, 0) and (3, 0), and the height of the fundamental rectangle is 14.
46. The hyperbola has vertices (0, −5) and (0, 5), and the width of the fundamental rectangle is 16.

In Exercises 47–50, graph each parabola.
47. a. \( y = x^2 \)
   b. \( y = x^2 − 5 \)
   c. \( y = (x + 2)^2 − 5 \)
48. a. \( y = −x^2 \)
   b. \( y = −x^2 + 3 \)
   c. \( y = −(x − 1)^2 + 3 \)
49. a. \( y = x^2 \)
   b. \( y = \frac{1}{2}x^2 \)
   c. \( y = −2x^2 \)
50. a. \( y = x^2 \)
   b. \( y = \frac{1}{3}x^2 \)
   c. \( y = −3x^2 \)

In Exercises 51–54, use the given graphs to solve each system of equations and each system of inequalities.
51. a. \( y = x^2 − 2x − 1 \)
   \( 2x + y = 3 \)
   b. \( y ≥ x^2 − 2x − 1 \)
   \( y ≤ −2x + 3 \)
52. a. \( x + y = −1 \)
   \( (x − 4)^2 + (y + 1)^2 = 40 \)
   b. \( y ≤ −x − 1 \)
   \( (x − 4)^2 + (y + 1)^2 ≤ 40 \)
53. a. \(x^2 + y^2 = 8\)
   \[y = x\]
   b. \(x^2 + y^2 \leq 8\)
   \[y \geq x\]

54. a. \(y = e^x\)
   \[y = -x + 1\]
   b. \(y \geq e^x\)
   \[y \leq -x + 1\]

Group discussion questions

55. Discovery Question A line and a circle can intersect at 0, 1, or 2 points. Illustrate each possibility with a sketch.

56. Discovery Question A circle and an ellipse can intersect at 0, 1, 2, 3, or 4 points. Illustrate each possibility with a sketch.

57. Discovery Question A circle and a hyperbola can intersect at 0, 1, 2, 3, or 4 points. Illustrate each possibility with a sketch.

58. Challenge Question Use a graphing calculator to graph each pair of functions.
   a. \(Y_1 = \sqrt{25 - x^2}, Y_2 = -Y_1\)  
      (Hint: Compare to \(x^2 + y^2 = 25\).)
   b. Split \(\frac{x^2}{25} + \frac{y^2}{16} = 1\) into \(Y_1\) and \(Y_2\), which represent upper and lower semiellipses. Then graph \(Y_1\) and \(Y_2\).

12.7 Cumulative Review

Solve each equation.
1. \(3(2x - 1) + 5 = 7(4x - 5) - 34\)
2. \(|2x - 1| = 25\)
3. \((2x - 1)^2 = 25\)
4. \(\frac{3x + 2}{2x - 7} = 4\)
5. \(\log(3x + 1) = 2\)

Chapter 12 Key Concepts

1. Methods for Solving 3 x 3 Systems of Linear Equations
   - Use the addition method and the substitution method to eliminate variables.
   - Use the augmented matrix method.

2. Augmented Matrix
   - A matrix is a rectangular array of numbers consisting of rows and columns.
   - A row consists of entries arranged horizontally.
   - A column consists of entries arranged vertically.
   - The dimension of a matrix is given by first stating the number of rows and then the number of columns.
   - An augmented matrix for a system of linear equations consists of the coefficients and constants in the equations.

   - The augmented matrix for \[\begin{align*}a_1x + b_1y + c_1z &= d_1 \\
a_2x + b_2y + c_2z &= d_2 \\
a_3x + b_3y + c_3z &= d_3\end{align*}\] is \[\begin{bmatrix} a_1 & b_1 & c_1 & | & d_1 \\ a_2 & b_2 & c_2 & | & d_2 \\ a_3 & b_3 & c_3 & | & d_3 \end{bmatrix}\].

3. Elementary Row Operations on Augmented Matrices
   - Any two rows in the matrix may be interchanged.
   - Any row in the matrix may be multiplied by a nonzero constant.
   - Any row in the matrix may be replaced by the sum of itself and a constant multiple of another row.

4. Reduced Row Echelon Form of a Matrix
   - The first nonzero entry in a row is 1. All other entries in the column containing the leading 1 are 0s.
• All nonzero rows are above any rows containing only 0s.
• The first nonzero entry in a row is to the left of the first nonzero entry in the following row.

5. Equation of a Plane  The graph of a linear equation of the form $Ax + By + Cz = D$ is a plane in three-dimensional space.

6. Basic Functions Examined in This Book Each type of function has an equation of a standard form and a graph with a characteristic shape.
• Linear functions: First-degree polynomial functions (The shape of the graph is a straight line.)
• Quadratic functions: Second-degree polynomial functions (The shape of the graph is a parabola.)
• Cubic functions: Third-degree polynomial functions
• Absolute value functions (The graph is V-shaped.)
• Square root functions
• Cube root functions
• Exponential functions
• Logarithmic functions
• Rational functions

7. Vertical and Horizontal Translations of $y = f(x)$
If $c$ is a positive real number,
• $y = f(x) + c$ shifts the graph of $y = f(x)$ up $c$ units.
• $y = f(x) - c$ shifts the graph of $y = f(x)$ down $c$ units.
• $y = f(x + c)$ shifts the graph of $y = f(x)$ left $c$ units.
• $y = f(x - c)$ shifts the graph of $y = f(x)$ right $c$ units.

8. Stretching, Shrinking, and Reflecting $y = f(x)$
• $y = -f(x)$ reflects the graph of $y = f(x)$ across the x-axis.
• $y = cf(x)$ stretches the graph of $y = f(x)$ vertically by a factor of $c$ for $c > 1$.
• $y = cf(x)$ shrinks the graph of $y = f(x)$ vertically by a factor of $c$ for $0 < c < 1$.

9. Operations on Functions If $f$ and $g$ are functions, then for all input values in both the domain of $f$ and $g$:
• Sum: $(f + g)(x) = f(x) + g(x)$
• Difference: $(f - g)(x) = f(x) - g(x)$
• Product: $(f \cdot g)(x) = f(x)g(x)$
• Quotient: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$ for $g(x) \neq 0$

10. Composite Function $f \circ g$
• If $f$ and $g$ are functions, then $(f \circ g)(x) = f(g(x))$ for all input values $x$ in the domain of $g$ for which $g(x)$ is an input value in the domain of $f$.
• If $f$ and $f^{-1}$ are inverses of each other, then $(f \circ f^{-1})(x) = x$ for each input value of $f^{-1}$ and $(f^{-1} \circ f)(x)$ for each input value of $f$.

11. Sequences
• A sequence is a function whose domain is a set of consecutive natural numbers.
• A finite sequence has a last term.
• An infinite sequence continues without end.
• A sequence is arithmetic if $a_n = a_{n-1} + d$; $d$ is called the common difference.
• The points of an arithmetic sequence lie on a line.
• A sequence is geometric if $a_n = ra_{n-1}$; $r$ is called the common ratio.
• The points of a geometric sequence lie on an exponential curve.

12. General Term of a Sequence
• A formula for $a_n$ is a formula for the general term of a sequence.
• A formula defining $a_n$ in terms of one or more of the preceding terms is called a recursive definition for $a_n$.
• The Fibonacci sequence defined by $a_1 = 1$, $a_2 = 1$, and $a_n = a_{n-2} + a_{n-1}$ is an example of a recursive definition.
• A formula for the general term of an arithmetic sequence is $a_n = a_1 + (n - 1)d$.
• A formula for the general term of a geometric sequence is $a_n = a_1r^{n-1}$.

13. Series
• A series is a sum of the terms of a sequence.
• Summation notation is used to denote a series: $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_{n-1} + a_n$
• The sum of the first $n$ terms of an arithmetic sequence is $S_n = \frac{n}{2}(a_1 + a_n)$.
• The sum of the first $n$ terms of a geometric sequence is $S_n = \frac{a_1(1 - r^n)}{1 - r}$.
• If $|r| < 1$, the infinite geometric series $S = \sum_{i=1}^{\infty} a_i$ is $S = \frac{a_1}{1 - r}$. If $|r| \geq 1$, this sum does not exist.

14. Distance and Midpoint Formulas For two points $(x_1, y_1)$ and $(x_2, y_2)$:
• Distance: $d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
• Midpoint: $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

15. Standard Forms for Conic Sections
• Circle: $(x - h)^2 + (y - k)^2 = r^2$
• Ellipse: $\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$ with horizontal major axis
• Hyperbola: $\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1$ opens horizontally
• Hyperbola: $\frac{(y - k)^2}{a^2} - \frac{(x - h)^2}{b^2} = 1$ opens vertically
• Parabola: $y - k = \frac{1}{4p}(x - h)^2$
Chapter 12
Review Exercises

Solving Systems of Linear Equations

1. Write an augmented matrix for each system of linear equations.
   a. \(2x - 5y = 17\)  
      \(3x + 4y = 14\)
   b. \(3x - 4y + 2z = -11\)
      \(2x + 2y + 3z = -3\)
      \(4x - y + 5z = -13\)

2. Write the solution for the system of linear equations represented by each augmented matrix.
   a. \(\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 6 \end{bmatrix}\)
   b. \(\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 8 \end{bmatrix}\)

3. Write the general solution and three particular solutions for the system of linear equations represented by this augmented matrix:
   \(\begin{bmatrix} 1 & 0 & 3 & 5 \\ 0 & 1 & -2 & 4 \\ 0 & 0 & 1 & 8 \end{bmatrix}\).

In Exercises 4–7, use the given elementary row operations to complete each matrix.

4. \(\begin{bmatrix} 3 & 5 & 13 \\ 1 & 4 & 2 \end{bmatrix}\) \(\overset{r_2 \leftrightarrow r_2}{\rightarrow}\) \(\begin{bmatrix} - & - & - \\ - & - & - \end{bmatrix}\)

5. \(\begin{bmatrix} 2 & 4 & -10 \\ 3 & 2 & -3 \end{bmatrix}\) \(\overset{r_1 \rightarrow 2r_1}{\rightarrow}\) \(\begin{bmatrix} - & - & - \\ 3 & 2 & -3 \end{bmatrix}\)

6. \(\begin{bmatrix} 1 & 2 & 14 \\ 3 & -1 & 13 \end{bmatrix}\) \(\overset{r_2 = -3r_1}{\rightarrow}\) \(\begin{bmatrix} 1 & 2 & 14 \\ - & - & - \end{bmatrix}\)

7. \(\begin{bmatrix} 1 & 3 & -2 & -6 \\ 0 & 1 & 2 & 3 \end{bmatrix}\) \(\overset{r_1 \rightarrow 2r_1}{\rightarrow}\) \(\begin{bmatrix} 0 & 1 & 2 & 3 \\ - & - & - \end{bmatrix}\)

In Exercises 8–14, solve each system of linear equations.

8. \(x + 2y = 17\)  
   \(2x + 5y = 41\)
9. \(3x - 2y = -16\)  
   \(x + 3y = 13\)
10. \(3x + 4y = -1\)  
    \(2x - 3y = 5\)  
    \(-3x + 15y = 10\)
11. \(x - 5y = 8\)  
    \(-3x + 15y = 10\)
12. \(x + 2y - z = -2\)  
    \(2x - y + z = 4\)  
    \(3x + 2y + 2z = 3\)
13. \(x - 3z = 10\)  
    \(2x + y = 6\)  
    \(2y + z = 5\)
14. \(2x - 3y + 4z = 11\)  
    \(3x + 2y - 4z = -4\)  
    \(4x + y - 3z = -1\)
15. Graph the plane defined by the linear equation \(5x + 2y + 2.5z = 10\).

16. Weights of Pallets of Bricks and Blocks A truck has an empty weight of 30,000 lb. On the first trip, the truck delivered two pallets of bricks and three pallets of concrete blocks. The gross weight on this delivery was 48,000 lb. On the second trip, the truck delivered four pallets of bricks and one pallet of concrete blocks. The gross weight on this delivery was 49,600 lb. Use this information to determine the weight of a pallet of bricks and the weight of a pallet of concrete blocks.

17. Price of Calculators A department store chain with three stores retails calculators of types A, B, and C. The table shows the number sold of each type of calculator and the total income from these sales at each store. Find the price of each type of calculator.

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>Total Sales ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Store 1</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>156</td>
</tr>
<tr>
<td>Store 2</td>
<td>2</td>
<td>4</td>
<td>8</td>
<td>294</td>
</tr>
<tr>
<td>Store 3</td>
<td>3</td>
<td>3</td>
<td>7</td>
<td>321</td>
</tr>
</tbody>
</table>

Translations of Graphs and Functions

18. Match each function with its graph.
   a. \(f(x) = x^2\)
   b. \(f(x) = x^2 - 5\)
   c. \(f(x) = x^2 + 5\)

19. Match each function with its graph.
   a. \(f(x) = \sqrt{x}\)
   b. \(f(x) = \sqrt{x - 4}\)
20. Match each function with its graph.
   a. \( f(x) = |x - 1| + 2 \)
   b. \( f(x) = |x - 2| + 1 \)
   c. \( f(x) = |x + 2| - 1 \)

21. Use the given graph of \( y = f(x) \) to graph each function.
   a. \( y = f(x - 3) \)
   b. \( y = f(x) + 4 \)
   c. \( y = f(x - 3) + 4 \)

22. Determine the vertex of each parabola by using the fact that the vertex of \( f(x) = x^2 \) is at \((0, 0)\).
   a. \( f(x) = x^2 + 7 \)
   b. \( f(x) = (x - 9)^2 \)
   c. \( f(x) = (x + 11)^2 - 12 \)

23. Use the given table of values to complete each table.

   \[
   \begin{array}{c|c}
   x & y = f(x) + 5 \\
   \hline
   -2 & 5 \\
   -1 & 9 \\
   0 & 3 \\
   1 & -4 \\
   2 & 0 \\
   \end{array}
   \]

24. The graph of each of these functions is a translation of the graph of \( y = f(x) \). Match each function to the correct description.
   a. \( y = f(x) - 17 \)  
   b. \( y = f(x - 17) \)  
   c. \( y = f(x) + 17 \)  
   d. \( y = f(x + 17) \)

25. Use the given table and match each function with its graph.

\[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
0 & 5 \\
1 & -3 \\
2 & 0 \\
3 & 2 \\
4 & -6 \\
\end{array}
\]

25a. \( y = f(x) + 2 \)  
25b. \( y = f(x + 2) \)  
25c. \( y = f(x - 1) + 3 \)  
25d. \( y = f(x + 3) - 1 \)

26. Use the graph of \( y = f(x) \) shown and match each function with its graph.

\[
\begin{array}{c|c}
\text{x} & \text{y} \\
\hline
-2 & 5 \\
-1 & 9 \\
0 & 3 \\
1 & -4 \\
2 & 0 \\
\end{array}
\]

26a. \( y = -f(x) \)  
26b. \( y = 2f(x) \)  
26c. \( y = -2f(x) \)  
26d. \( y = \frac{1}{2}f(x) \)
27. Use the graph of \( y = f(x) \) shown and match each function with its graph.

C. \( y = -f(x) \)  
D. \( y = \frac{1}{3}f(x) \)

29. Match each function with the description that compares its graph to the graph of \( y = f(x) \).

a. \( y = 2f(x) \)  
b. \( y = -f(x) \)  
c. \( y = f(x + 2) \)  
d. \( y = f(x) + 2 \)

Algebra of Functions

In Exercises 30–32, evaluate each expression, given \( f(x) = 3x - 7 \) and \( g(x) = 2x^2 + 1 \).

30. a. \( f(10) \)  
b. \( g(10) \)  
c. \( (f + g)(10) \)  
d. \( (f - g)(10) \)

31. a. \( (f \cdot g)(2) \)  
b. \( \left( \frac{f}{g} \right)(2) \)  
c. \( \left( \frac{g}{f} \right)(2) \)  
d. \( (g - f)(2) \)

32. a. \( (f \circ g)(2) \)  
b. \( (g \circ f)(2) \)  
c. \( (f \cdot g)(2) \)  
d. \( (g \circ g)(2) \)

33. Use the given tables for \( f \) and \( g \) to form a table of values for each function.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
<th>( x )</th>
<th>( g(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-7</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>1</td>
<td>-4</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>-1</td>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>3</td>
<td>19</td>
</tr>
</tbody>
</table>

a. \( f + g \)  
b. \( f - g \)  
c. \( f \cdot g \)  
d. \( \frac{f}{g} \)
34. Use the given graphs for \( f \) and \( g \) to form a set of ordered pairs for each function.

\[
\begin{array}{c|c|c|c|c}
 x & f(x) & x & g(x) \\
\hline
 0 & 2 & 2 & 0 \\
 1 & 3 & 2 & 1 \\
 2 & 4 & 3 & 2 \\
 3 & 5 & 4 & 3 \\
 4 & 6 & 5 & 4 \\
\end{array}
\]

35. Use the given tables for \( f \) and \( g \) to complete a table for \( f \circ g \).

\[
\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c}
 x & f(x) & x & g(x) & f \circ g(x) \\
\hline
 0 & 2 & 2 & 0 & 0 \\
 1 & 3 & 2 & 1 & 1 \\
 2 & 4 & 3 & 2 & 2 \\
 3 & 5 & 4 & 3 & 3 \\
 4 & 6 & 5 & 4 & 4 \\
\end{array}
\]

36. Use the given graphs of \( f \) and \( g \) to graph \( f + g \).

42. **Composing Cost and Production Functions** The number of desks a factory can produce weekly is a function of the number of hours \( t \) it operates. This function is \( N(t) = 3t \) for \( 0 \leq t \leq 168 \). The cost of manufacturing \( N \) desks is given by \( C(N) = \left( 150 - \frac{N}{4} \right) N \).

43. Write the first six terms of an arithmetic sequence that satisfies the given conditions.

\( a_1 = 5, \quad d = 4 \)  
\( a_1 = 5, \quad d = -4 \)
\( a_1 = 5, \quad a_2 = 8 \)
\( a_1 = 5, \quad a_6 = 20 \)

44. Write the first six terms of a geometric sequence that satisfies the given conditions.

\( a_1 = 3, \quad r = 2 \)
\( a_1 = 3, \quad r = -2 \)
\( a_1 = 3, \quad a_2 = 12 \)
\( a_1 = 1, \quad a_6 = 243 \)

45. Write the first five terms of each sequence.

\( a_n = 2n + 7 \)
\( a_n = n^2 + 3 \)
\( a_n = 64 \left( \frac{1}{2} \right)^n \)
\( a_1 = 3, \quad a_n = a_{n-1} + 7 \) \( \text{for } n > 1 \)

46. An arithmetic sequence has \( a_1 = 11 \) and \( d = 3 \). Find \( a_{101} \).

47. A geometric sequence has \( a_1 = \frac{1}{64} \) and \( r = 2 \). Find \( a_{11} \).

48. An arithmetic sequence has \( a_1 = -20 \), \( a_n = 300 \), and \( d = 8 \). Find \( n \).

49. A geometric sequence has \( a_1 = \frac{1}{15625} \), \( a_n = 78,125 \), and \( \alpha = 5 \). Find \( n \).

50. Write the terms of each series, and then add these terms.

\( \sum_{i=1}^{6} (3i - 2) \)
\( \sum_{k=3}^{7} (k^2 - 2k) \)
\( \sum_{j=1}^{5} j^3 \)
\( \sum_{i=1}^{5} 10 \)
51. An arithmetic sequence has \( a_1 = 23 \) and \( a_{50} = 366 \). Find \( S_{50} \), the sum of the first 50 terms.

52. Write 0.121212 \ldots as a fraction.

53. Evaluate \( \sum_{k=1}^{\infty} \frac{3^k}{5^k} \).

54. **Rungs of a Ladder** The lengths of the rungs of a wooden ladder form an arithmetic sequence. There are 17 rungs ranging in length from 80 to 46 cm. Find the total length of these 17 rungs.

![Image of a ladder with rungs]

55. **Perimeter of an Art Design** An art design is formed by drawing a square with sides of 20 cm and then connecting the midpoints of the sides to form a second square. (See the figure.) If this process is continued infinitely, determine the perimeter of the third square and the total perimeter of all the squares.

![Image of an art design]

**Conic Sections**

56. Determine the distance from \((-4, 7)\) to \((1, -5)\).

57. Determine the midpoint between \((-5, 2)\) and \((7, 12)\).

In Exercises 58–60, refer to the following figure.

![Image of a graphed circle]

58. Determine the center of the graphed circle.

59. Determine the length of a radius of the graphed circle.

60. Determine the equation of the graphed circle.

In Exercises 61–66, match each graph with the corresponding equation.

61. \( y = 2x - 4 \)

62. \( y = 2(x - 1)^2 \)

63. \( (x - 1)^2 + (y + 2)^2 = 4 \)

64. \( \frac{(x - 1)^2}{9} + \frac{(y + 2)^2}{4} = 1 \)

65. \( \frac{(x - 1)^2}{9} - \frac{(y + 2)^2}{4} = 1 \)

66. \( y = \sqrt{4 - x^2} \)

![Graphs A, B, C, D, E, F]

In Exercises 67–76, graph each equation.

67. \( y = \frac{1}{2}x^2 - 3 \)

68. \( y = \frac{1}{2}x - 3 \)

69. \( y = -x^2 + 4 \)

70. \( x^2 + y^2 = 36 \)

71. \( (x + 3)^2 + (y - 2)^2 = 9 \)

72. \( y = \sqrt{25 - x^2} \)

73. \( y = -\sqrt{25 - x^2} \)

74. \( \frac{x^2}{49} + \frac{y^2}{16} = 1 \)

75. \( \frac{(x - 1)^2}{4} + \frac{(y + 3)^2}{9} = 1 \)

76. \( \frac{x^2}{36} - \frac{y^2}{9} = 1 \)

In Exercises 77 and 78, refer to the following figure.

![Graphs G and H]
77. Use the graphs shown in the figure to solve this system of equations.

\[
y = \frac{1}{2}x^2 + 1
\]

\[
\frac{x^2}{4} + \frac{(y - 3)^2}{9} = 1
\]

78. Use the graphs shown in the figure to graph the solution of this system of inequalities.

\[
y \geq \frac{1}{2}x^2 + 1
\]

\[
\frac{x^2}{4} \geq \frac{(y - 3)^2}{9}
\]
Objective 12.5.1 Add, Subtract, Multiply, and Divide Two Functions

7. Given \( f(x) = 3x - 5 \) and \( g(x) = x + 2 \), determine each function.
   a. \( f + g \)  
   b. \( f - g \)
   c. \( f \cdot g \)  
   d. \( \frac{f}{g} \)

Objective 12.5.2 Form the Composition of Two Functions

8. Given \( f(x) = x^2 + 1 \) and \( g(x) = 2x - 1 \), determine
   a. \( (f \circ g)(5) \)  
   b. \( (g \circ f)(5) \)
   c. \( (f \circ g)(x) \)  
   d. \( (g \circ f)(x) \)

Objective 12.4.2 Use Stretching and Shrinking Factors to Graph Functions

6. Use the graph of \( y = f(x) \) shown and match each function with its graph.
   a. \( y = 2f(x) \)  
   b. \( y = -2f(x) \)
   c. \( y = \frac{1}{2} f(x) \)  
   d. \( y = -\frac{1}{2} f(x) \)

Objective 12.6.3 Evaluate an Infinite Geometric Series

11. a. Evaluate \( \sum_{i=1}^{\infty} \left( \frac{1}{2} \right)^i \).
    b. Write 0.888... as a fraction.
    c. Write 0.545454... as a fraction.

Objective 12.6.2 Use Summation Notation and Evaluate the Series Associated with a Finite Sequence

10. Evaluate each series.
   a. The sum of the first five terms of an arithmetic sequence with \( a_1 = 2 \) and \( d = 5 \)
   b. The sum of the first five terms of a geometric sequence with \( a_1 = 2 \) and \( r = 3 \)
   c. The sum of the first 100 terms of an arithmetic sequence with \( a_1 = 10 \) and \( a_{100} = 21 \)
   d. The sum of the first 21 terms of a geometric sequence with \( a_1 = 6 \) and \( r = -2 \)
   e. \( \sum_{i=1}^{6} (3i^2 + i) \)

Objective 12.6.1 Calculate the Terms of Arithmetic and Geometric Sequences

9. a. Write the first six terms of an arithmetic sequence with \( a_1 = 4 \) and \( d = 3 \).
   b. Write the first six terms of a geometric sequence with \( a_1 = 4 \) and \( r = 3 \).
   c. Write the first six terms of an arithmetic sequence with \( a_1 = 2 \) and \( a_2 = 8 \).
   d. Write the first six terms of a geometric sequence with \( a_1 = 2 \) and \( a_2 = 8 \).

Objective 12.7.1 Graph the Conic Sections and Write their Equations in Standard Form

12. Graph each of these conic sections.
   a. \( y = (x + 3)^2 - 4 \)
   b. \( (x - 3)^2 + (y + 4)^2 = 16 \)
   c. \( \frac{(x - 3)^2}{16} + \frac{(y + 4)^2}{25} = 1 \)
   d. \( \frac{x^2}{25} - \frac{y^2}{16} = 1 \)
13. Write the equation for each of these conic sections.

a. 

b. 

c. 

d. 